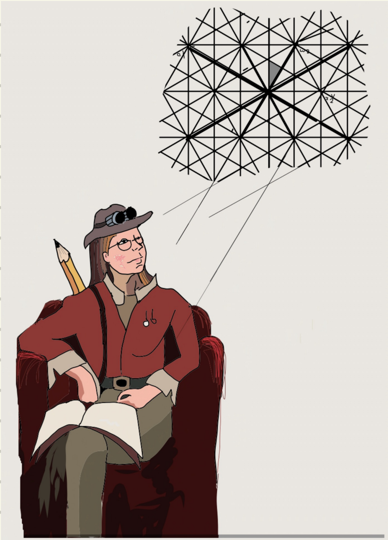
Affine Gordon-Bender-Knuth identities and cylindric Young tableaux



AlCoVE 2022

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Outline

① Basic definitions & Robinson-Schensted algorithm

② Standard Young tableaux of bounded height

③ Noncrossing and nonnesting involutions

④ Cylindric SYT

⑤ Original Motivation

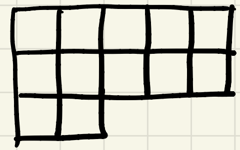
⑥ Affine Gordon-Bender-Knuth identities

Basic definitions

Def) $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n if

$$\lambda_1 \geq \cdots \geq \lambda_k > 0, \qquad \lambda_1 + \cdots + \lambda_k = n.$$

length of $\lambda = \ell(\lambda) = k = \#$ parts

The Young diagram of λ is  $\lambda = (5,5,2)$.

A standard Young tableau of shape λ is
$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 7 \\ \hline 3 & 4 & 8 & \\ \cline{1-3} 5 & & & \\ \cline{1-1} \end{array}$$

A semistandard Young tableau "
(column-strict tableau)
$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 & \\ \cline{1-3} 4 & & & \\ \cline{1-1} \end{array}$$

A row-strict tableau "
$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 4 & \\ \cline{1-3} 2 & & & \\ \cline{1-1} \end{array}$$

Robinson–Schensted algorithm

π : permutation of $[n] = \{1,2,\dots,n\}$.

$\updownarrow$ 1-1

(P,Q) : pair of SYTs of size n and of same shape

ex) $$\pi = 4\,1\,5\,6\,3\,2 \;\xleftrightarrow{\text{RS}}\; P = \begin{array}{lll}1&2&6\\3&5\\4\end{array} \qquad Q = \begin{array}{lll}1&3&4\\2&5\\6\end{array}$$

* insertion algorithm

$$P:\quad \emptyset \leftarrow 4 \qquad 4 \leftarrow 1 \qquad \begin{array}{l}1\\4\end{array} \leftarrow 5 \qquad \begin{array}{ll}1&5\\4\end{array} \leftarrow 6 \qquad \begin{array}{lll}1&5&6\\4\end{array} \leftarrow 3 \qquad \begin{array}{lll}1&3&6\\4&5\end{array} \leftarrow 2 \qquad \boxed{\begin{array}{lll}1&2&6\\3&5\\4\end{array}}$$

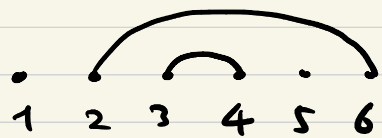
$$Q:\quad \emptyset \qquad 1 \qquad \begin{array}{l}1\\2\end{array} \qquad \begin{array}{ll}1&3\\2\end{array} \qquad \begin{array}{lll}1&3&4\\2\end{array} \qquad \begin{array}{lll}1&3&4\\2&5\end{array} \qquad \boxed{\begin{array}{lll}1&3&4\\2&5\\6\end{array}}$$

Consequences of RS algorithm

- If $\pi \xleftrightarrow{RS} (P,Q)$, then $\pi^{-1} \xleftrightarrow{RS} (Q,P)$.

- π : involutions of $[n]$ $\xleftrightarrow{RS}$ SYTs of size n. (P,P).
  $(\pi^2 = id)$

- $\pi = (26)(34)$



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 4 & 3 & 5 & 2 \end{pmatrix} \xleftrightarrow{RS} \begin{array}{lll} 1 & 2 & 5 \\ 3 & & \\ 4 & & \\ 6 & & \end{array}$$

involution $\longleftrightarrow$ matching

SYTs with bounded height.

Def) SYT_n = set of SYTs of size n.

$$\text{ht}=3\left\{\begin{array}{|c|c|c|c|}\hline 1&2&6&7\\\hline 3&4&8\\\cline{1-3} 5\\\cline{1-1}\end{array}\right.$$

$$SYT_n(h) = \{ T \in SYT_n : \underbrace{ht(T)}_{=\#\text{ rows}} \le h \}$$

ex). $|SYT_n(2)| = \binom{n}{\lfloor n/2 \rfloor}$

$|SYT_n(3)|$ = # Motzkin paths of length n.

$|SYT_n(4)| = C_{\lfloor \frac{n+1}{2} \rfloor} C_{\lceil \frac{n+1}{2} \rceil}$, $\quad C_n = \frac{1}{n+1}\binom{2n}{n}$ = nth Catalan number.

<u>Thm</u> (Gessel. 1990)

$$\sum_{n\ge 0} |SYT_n(2h+1)| \frac{x^n}{n!} = \exp(x) \det\left(I_{-i+j}(2x) - I_{i+j}(2x) \right)_{i,j=1}^{h}$$

$$I_\alpha(2x) = \sum_{\ell \ge 0} \frac{x^{2\ell+|\alpha|}}{\ell!\,(\ell+|\alpha|)!} \quad \text{(Modified Bessel function)}$$

Properties of RS-algorithm

① If $\pi \xleftrightarrow{RS} (P,Q)$, then $\pi^{-1} \xleftrightarrow{RS} (Q,P)$.

② max length of decreasing subsequence of π = $ht(P)$.

③ 〃 Increasing 〃 = $width(P)$

ex) $\pi = 4156327 \xleftrightarrow{RS}$

P	Q
1267	1347
35	25
4	6

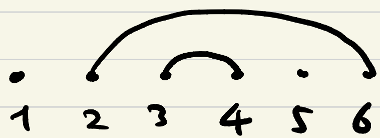
max dec 432

max inc 4567

Consequences of RS algorithm

π : involutions of $[n]$ $\xleftrightarrow{RS}$ SYTs of size n. (P,P).
$(\pi^2 = id)$

$\pi = (26)(34)$



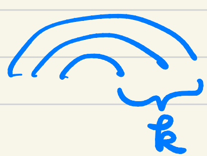
$\xleftrightarrow{RS}$

$$\begin{array}{lll} 1 & 2 & 5 \\ 3 & & \\ 4 & & \\ 6 & & \end{array}$$

$$\| $$

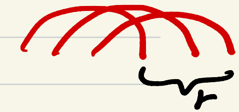
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 4 & 3 & 5 & 2 \end{pmatrix}$$

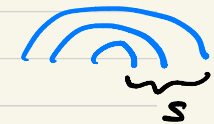
involutions of $[n]$ with no k-nesting $\xleftrightarrow{RS}$ SYTs of size n with height $< 2k$



$\leftrightarrow$ dec seq of length $2k$

r-noncrossing and s-nonnesting involutions

Def) An involution π is r-noncrossing if π has no 

" s-nonnesting " 

$NCNN_n(r,s)$ = set of r-noncrossing and s-nonnesting involutions of $[n]$

<u>Thm</u> (Chen, Deng, Du, Stanley, Yan. 2007)

$$\# NCNN_n(r,s) = \# NCNN_n(s,r).$$

In particular,

\# k-noncrossing involutions of $[n]$

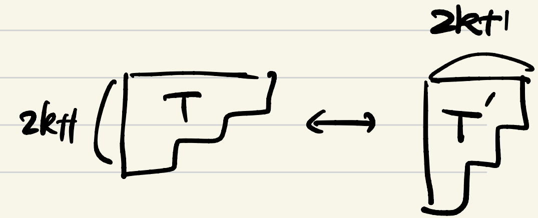
= \# k-nonnesting " .

SYTs of size n with height $\leq 2k+1$

$\overset{\text{RS}}{=}$ # $(k+1)$-nonnesting involutions of $[n]$

$\overset{\text{CDDSY}}{=}$ # $(k+1)$-noncrossing "

By taking transpose,



SYTs of size n with height $\leq 2k+1$

$=$ # SYTs of size n with width $\leq 2k+1$

Question

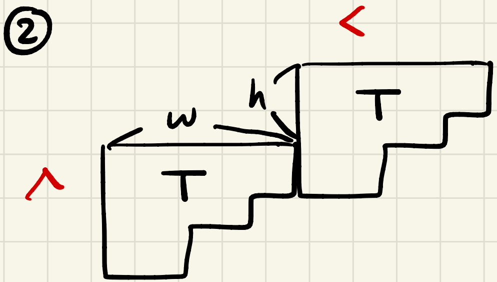
SYTs of size n with height $\leq 2h+1$ and width $\leq 2w+1$

$\overset{?}{=}$ # $(h+1)$-nonnesting and $(w+1)$-noncrossing involutions of $[n]$

Def) (w,h)-cylindric SYT is an SYT T such that

① width of $T \le w$

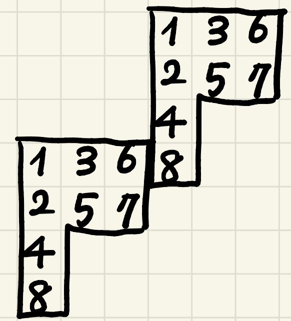
②



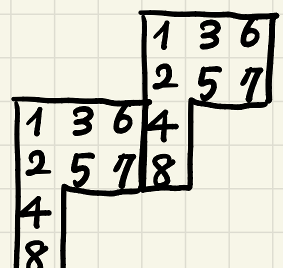
← also increases along each row & column.

valid skew shape

ex) $\begin{array}{|ccc|} 1 & 3 & 6 \\ 2 & 5 & 7 \\ 4 & & \\ 8 & & \end{array}$ is (3,3)-cylindric SYT :



But not (3,2)-cylindric :



Let $CSYT_n(w,h) = \{ (w,h)\text{-cylindric SYTs of size } n \}$

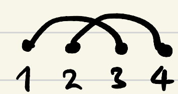
<u>Thm</u> (Huh, Kim, Krattenthaler, Okada)

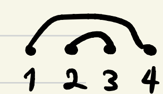
$$\# CSYT_n(2w+1, 2h+1) = \# NCNN_n(w+1, h+1)$$

ex) $n=4$, $h=1$, $w=1$.

There are 2 SYTs not counted in LHS : $1\,2\,3\,4$ $\quad \begin{matrix}1\\2\\3\\4\end{matrix}$

" 2 involutions " RHS :

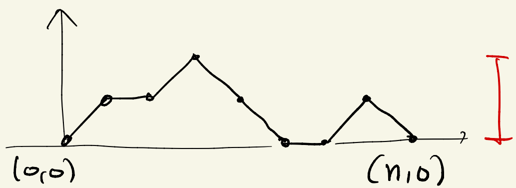




Open Problem : Find a bijective proof.

Original Motivation

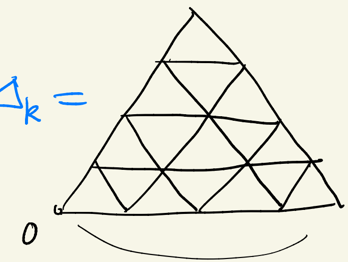
Def) Motzkin path



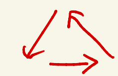
height $= 2$

Def) Triangle lattice of size k

$\Delta_k =$



steps :

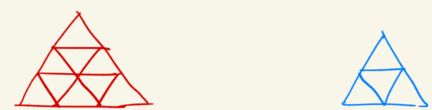


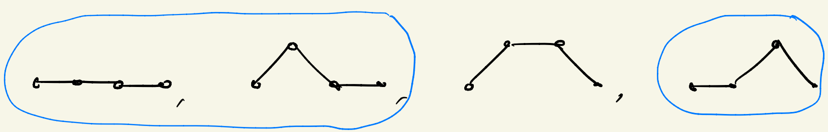
<u>Thm (Mortimer & Prellberg, 2015)</u>

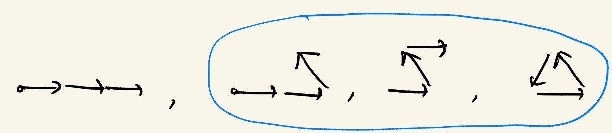
Motzkin paths of length n with height $\le h$ (without horizontal steps of height h)

$=$ # lattice walks of length n from 0 in Δ_{2h+1} (Δ_{2h})

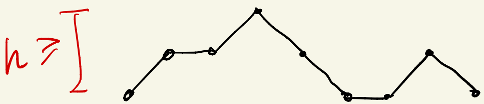
<u>ex)</u> $n=3$, $h=1$

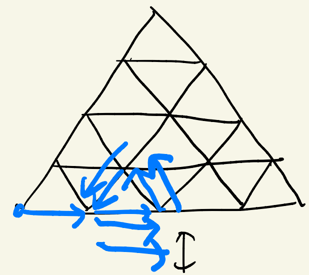






Motzkin path $ht \leq h$ $\longleftrightarrow$ paths in Δ_{2h+1}





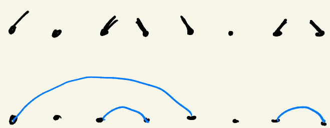
→ : add a cell in col 1

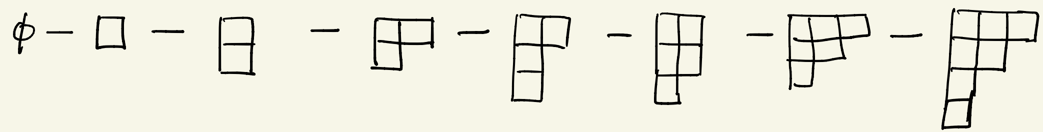
↖ : " 2

↙ : " 3

$\updownarrow$

involution





$\updownarrow$

1	3	6
2	5	
4		
7		

$(3, 2h+1)$-cylindric SYT

no 2-crossing

no $(h+1)$-nesting

Mortimer & Prellberg's result is equivalent to

$$NCNN_n(2, h+1) = CSYT_n(3, 2h+1)$$

\# 2-noncrossing $(h+1)$-nonnesting involutions on $\{1, \dots, n\}$

\# $(3, 2h+1)$-cylindric SYTs of size n.

This is a special case of our theorem:

Thm (Huh, Kim, Krattenthaler, Okada)

$$NCNN_n(w+1, h+1) = CSYT_n(2w+1, 2h+1).$$

Note: Courtiel, Elvey Price, Marcovici found a bijective proof of M-P result.

Thm (Huh, Kim, Krattenthaler, Okada)

$$\# \mathrm{CSYT}_n(2w+1, 2h+1) = \# \mathrm{NCNN}_n(w+1, h+1).$$

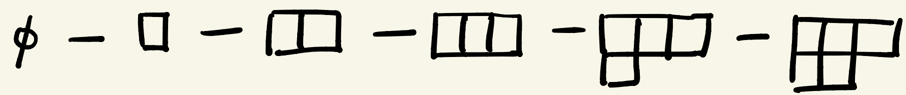
Idea of Proof

① Express each side as # lattice walks

② Express # lattice walks using determinants

③ Prove det = det.

* Lattice walks for $\mathrm{CSYT}_n(2w+1, 2h+1)$.

CSYT $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$ $\longleftrightarrow$ 

$\longleftrightarrow (0,0,0) - (1,0,0) - (2,0,0) - (3,0,0) - (3,1,0) - (3,2,0)$

a walk in the region $\{(x_1, x_2, x_3) : x_1 \geq x_2 \geq x_3 \geq 0\}$

with step set $\varepsilon_1 = (1,0,0)$, $\varepsilon_2 = (0,1,0)$, $\varepsilon_3 = (0,0,1)$.

Under this correspondence

$\mathrm{CSYT}_n(2w+1, 2h+1) \longleftrightarrow$ walks from 0 of length n in

$$\{(x_1, \dots, x_{2h+1}) : x_1 \geq \cdots \geq x_{2h+1} \geq 0,\ x_1 - x_{2h+1} \leq 2w+1\}$$

$$= \{(x_1, \dots, x_{2h+1}) : x_1 \geq \cdots \geq x_{2h+1} \geq x_1 - 2w - 1\}.$$

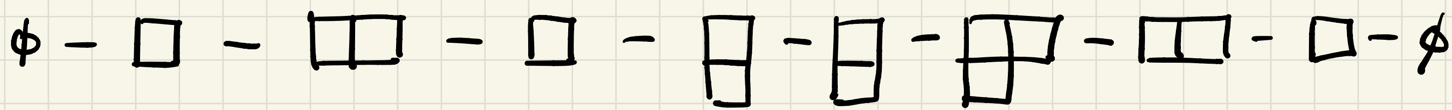
alcove of affine Weyl group of type $\tilde{A}_{2h}$

Def) A *vacillating tableau* is a sequence of partitions

$$\phi = \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(n)} = \phi \quad \text{such that}$$

$\lambda^{(i)}$ and $\lambda^{(i+1)}$ differ by at most one cell.

ex)

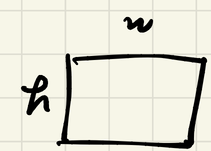


<u>Prop</u>

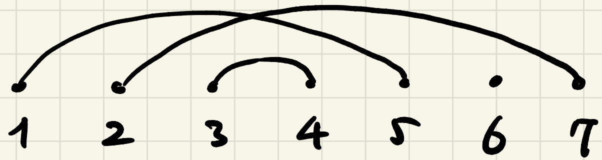
$$\text{involutions on } [n] \overset{1-1}{\longleftrightarrow} \text{vacillating tableaux of length } n$$

$$NCNN_n(w+1, h+1) \overset{1-1}{\longleftrightarrow} \quad \text{"}$$

with every partition contained in



ex)



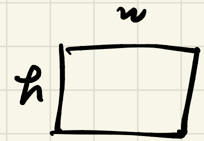
$$\begin{array}{ccccccccccccccc} & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 \\ \longleftrightarrow & \emptyset & - & \boxed{1} & - & \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} & - & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} & - & \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} & - & \boxed{2} & - & \boxed{2} & - & \emptyset \\ \longleftrightarrow & \emptyset & - & \square & - & \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} & - & \begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array} & - & \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} & - & \square & - & \square & - & \emptyset \end{array}$$

- Proceed from right to left.
- In vertex i, if $\underset{i}{\bullet}$ then $\lambda^{(i)} = \lambda^{(i+1)}$

  if $\underset{j}{\bullet}\frown\underset{i}{\bullet}$ then $\lambda^{(i)} = \lambda^{(i+1)} \leftarrow j$

  if $\underset{i}{\bullet}\frown\underset{j}{\bullet}$ then $\lambda^{(i)} = \lambda^{(i+1)} - \{i\}$.

$NCNN_n(w+1, h+1) \overset{1-1}{\longleftrightarrow}$ vacillating tableaux of length n with every partition contained in



$\longleftrightarrow$ lattice walks from 0 to 0 of length n in region $\{(x_1, \dots, x_n) : w \geqslant x_1 \geqslant \cdots \geqslant x_n \geqslant 0\}$ with step set $\{\pm \varepsilon_i\} \cup \{0\}$.

alcove of affine Weyl group of type $\tilde{C}_n$

Filaseta (1985) computed # lattice walks in alcove of affine Weyl groups.

Applying Filaseta's result, we get

$$\left|CSYT_n(w,h)\right| = \sum_{\substack{\lambda \vdash n \\ \lambda_1 - \lambda_h \leq w}} \sum_{\substack{k_1 + \cdots + k_h = 0 \\ k_1, \ldots, k_h \in \mathbb{Z}}} n! \det\left(\frac{1}{(\lambda_i - i + j + (w+h)k_i)!}\right)_{i,j=1}^{h}$$

and

$$\sum_{n \geq 0} \left|NCNN_n(w+1,h+1)\right| \frac{x^n}{n!}$$

$$= \exp(x) \sum_{k_1, \ldots, k_h \in \mathbb{Z}} \det\left(I_{-i+j+(2h+2w+2)k_i}(2x) - I_{i+j+(2h+2w+2)k_i}(2x)\right)_{i,j=1}^{h}$$

where $I_k(2x) = \sum_{\ell \geq 0} \frac{x^{2\ell+k}}{\ell!(\ell+k)!}$ (Modified Bessel ftn).

So, to prove our theorem

$$|CSYT_n(2w+1, 2h+1)| = |NCNN_n(w+1, h+1)|$$

it suffices to prove

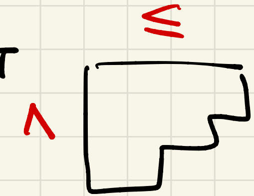
$$\sum_{\lambda:\, \lambda_1 - \lambda_n \leq w} \ \sum_{\substack{k_1 + \cdots + k_h = 0 \\ k_1, \ldots, k_n \in \mathbb{Z}}} x^n \det\left(\frac{1}{(\lambda_i - i + j + (w+h)k_i)!} \right)_{i,j=1}^{h}$$

$$= \exp(x) \sum_{k_1, \ldots, k_h \in \mathbb{Z}} \det\left(I_{-i+j+(2h+2w+2)k_i}(2x) - I_{i+j+(2h+2w+2)k_i}(2x) \right)_{i,j=1}^{h}$$

We will show a symmetric function generalization of this.

Def) Schur function

Recall SSYT



$$s_\lambda = \sum_{T \in SSYT(\lambda)} x_1^{\#1's}\, x_2^{\#2's} \cdots$$

ex) $\lambda = (2,1)$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \cdots + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} + \cdots$$

$$s_\lambda = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + \cdots + x_1 x_2 x_3 + x_1 x_2 x_3 + \cdots$$

Note: $[x_1 x_2 \cdots x_n]\, s_\lambda = \#\, SYT(\lambda)$, where $|\lambda| = n$.

In the above example $[x_1 x_2 x_3]\, s_{(2,1)} = 2$

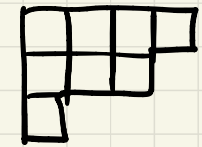
Thm (Jacobi-Trudi formula)

If $\lambda = (\lambda_1, \dots, \lambda_k)$, then

$$s_\lambda = \det\left(h_{\lambda_i - i + j} \right)_{i,j=1}^{k} .$$

$$s_{\lambda'} = \det\left(e_{\lambda_i - i + j} \right)_{i,j=1}^{k} .$$

λ' : transpose of λ

$\lambda =$ 

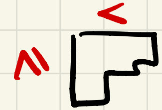
$\lambda' =$ 

Thm (Gordon-Bender-Knuth, 1972)

$$\sum_{\ell(\lambda) \leq 2m+1} s_{\lambda'} = \sum_{k \geq 0} e_k \det\left(f_{-i+j} - f_{i+j} \right)_{i,j=1}^{m}$$

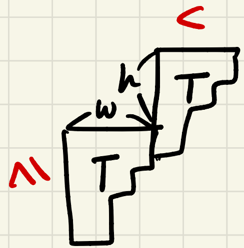
$$\sum_{\ell(\lambda) \leq 2m} s_{\lambda'} = \det\left(f_{-i+j} + f_{i+j-1} \right)_{i,j=1}^{m}$$

where $f_k = \sum_{n \geq 0} e_n e_{n+k}$.

Def) A row-strict tableau is . (transpose of SSYT)

Def) (w,h)-cylindric RST is an RST T such that

width $\leq w$ and



← This is also an RST.

Def) $CRST_\lambda(w,h) = \{ T : (w,h)\text{-cylindric RST of shape } \lambda \}$

<u>Thm</u> (Jacobi-Trudi for cylindric Schur ftn)

$$\sum_{T \in CRST_\lambda(w,h)} x_T = \sum_{\substack{k_1+\cdots+k_h=0 \\ k_1,\ldots,k_h \in \mathbb{Z}}} \det\left(e_{\lambda_i - i + j + (h+w)k_i}\right)_{i,j=1}^{h}$$

Pf) Follows from Gessel-Krattenthaler 1997.

Thm (Gordon-Bender-Knuth, 1972)

$$\sum_{T \in RST(2h+1)} x_T = \sum_{k \geq 0} e_k \det\left(f_{-i+j} - f_{i+j}\right)_{i,j=1}^{h}$$

$$\sum_{T \in RST(2h)} x_T = \det\left(f_{-i+j} + f_{i+j-1}\right)_{i,j=1}^{h}$$

where $f_k = \sum_{n \geq 0} e_n e_{n+k}$.

Thm (Affine Gordon-Bender-Knuth)

$$\sum_{T \in CRST(w,2h+1)} x_T = \sum_{k \geq 0} e_k \sum_{k_1,\ldots,k_h \in \mathbb{Z}} \det\left(f_{-i+j+(w+2h+1)k_i} - f_{i+j+(w+2h+1)k_i}\right)_{i,j=1}^{h}$$

$$\sum_{T \in CRST(w,2h)} x_T = \sum_{k_1,\ldots,k_h \in \mathbb{Z}} (-1)^{k_1+\cdots+k_h} \det\left(f_{-i+j+(w+2h)k_i} + f_{i+j+(w+2h)k_i}\right)_{i,j=1}^{h}$$

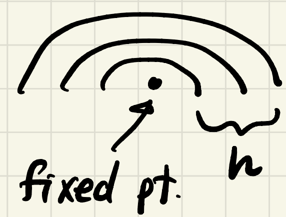
Cor

① $\# \, CSYT_n(2w+1, 2h+1) = \# \, NCNN_n(w+1, h+1)$

② $\# \, CSYT_n(2w+1, 2h) = \# \, NCNN_n(w+1, h+\tfrac{1}{2})$

③ $\# \, CSYT_n(2w, 2h+1) = \# \, NCNN_n(w+\tfrac{1}{2}, h+\tfrac{1}{2})$

④ $\# \, CSYT_n(2w, 2h) = \sum_{M \in NCNN_n'(w+1, h+1)} (-1)^{fix_1(M)}.$

$(h+\tfrac{1}{2})$-nonnesting $\Longleftrightarrow$ no 

If $w=1$, then ①, ② reduce to Mortimer–Prellberg result.

If $\omega=1$, then ③, ④ become

<u>Cor</u>

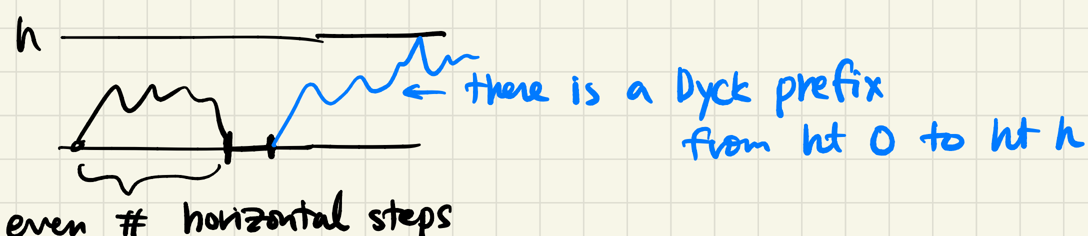
③ # Dyck prefixes of length n with height $\leq 2h+1$

= # Motzkin paths of length n with height $\leq h$ and every horizontal step is on x-axis.

④ # Dyck prefixes of length n with height $\leq 2h$

= # Motzkin paths of length n with height $\leq h$ and every horizontal step is on x-axis.

s.t.



We found bijective proofs using recent results of Gu–Prodinger and Dershowitz.

Thank You

for your attention !