HOMEWORK

Contents

1. Homework 1 (Due: Sep 21)

Problem 1.1. Let $\mathcal{L}$ be a positive-definite linear functional with monic OPS $\{P_n(x)\}_{n\geq 0}$. Prove the following extremal property: for any monic real polynomial $\pi(x) \neq P_n(x)$ of degree n,

$$\mathcal{L}(P_n(x)^2) < \mathcal{L}(\pi(x)^2).$$

Solution. Let $\pi(x) = \sum_{k=0}^n a_k P_k(x)$. Since both $\pi(x)$ and $P_n(x)$ are monic, we have $a_n = 1$ **[3 points]**. Then

$$\begin{aligned}\mathcal{L}(\pi(x)^2) &= \sum_{k=0}^n a_k^2 \mathcal{L}(P_k(x)^2) \quad \textbf{[4 points]}\\ &\geq a_n^2 \mathcal{L}(P_n(x)^2) = \mathcal{L}(P_n(x)^2) \quad \textbf{[3 points]}.\end{aligned}$$

□

Problem 1.2. Let $\mathcal{L}$ be a linear functional such that $\Delta_n \neq 0$ for all $n \geq 0$. Prove that if $\pi(x)$ is a polynomial such that $\mathcal{L}(x^k \pi(x)) = 0$ for all $k \geq 0$, then $\pi(x) = 0$.

Solution. Since $\Delta_n \neq 0$, there is a monic OPS $\{P_n(x)\}_{n\geq 0}$ for $\mathcal{L}$ **[3 points]**. Let $\pi(x) = \sum_{k=0}^n a_k P_k(x)$. Since $\mathcal{L}(x^k \pi(x)) = 0$ for all $k \geq 0$, we have $\mathcal{L}(p(x)\pi(x)) = 0$ for any polynomial $p(x)$ **[3 points]**. Then, for each $0 \leq k \leq n$, we have $0 = \mathcal{L}(P_k(x)\pi(x)) = a_k \mathcal{L}(P_k(x)^2)$ **[2 points]**. Since $\mathcal{L}(P_k(x)^2) \neq 0$, we get $a_k = 0$ for all $0 \leq k \leq n$ **[2 points]**. Hence $\pi(x) = 0$.

A common mistake: It is not true in general that $\mathcal{L}(x^k P_n(x)) = 0$ for $k \neq n$. We can only say that $\mathcal{L}(x^k P_n(x)) = 0$ for $k < n$. □

Problem 1.3. The *Tchebyshev polynomials of the second kind* $U_n(x)$ are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta, \qquad n \geq 0.$$

(1) Prove that $U_n(x)$ is a polynomial of degree n.
(2) Prove that

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \qquad n \geq 0,$$

where $U_{-1}(x) = 0$ and $U_0(x) = 1$.
(3) Prove that

$$\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{1/2}dx = \frac{\pi}{2}\delta_{m,n}.$$

(4) Find the 3-term recurrence for the normalized Tchebyshev polynomials of the second kind. More precisely, find the numbers b_n and λ_n such that

$$\hat{U}_{n+1}(x) = (x-b_n)\hat{U}_n(x) - \lambda_n \hat{U}_{n-1}(x), \qquad n \geq 0,$$

where $\hat{U}_n(x)$ is the monic polynomial that is a scalar multiple of $U_n(x)$.

Solution. (1) This follows from (2) **[2 points]**.

(2) By the addition rule for the sine function,

$$\begin{aligned}\sin(n+1)\theta &= \sin n\theta\cos\theta + \cos n\theta \sin\theta,\\ \sin(n-1)\theta &= \sin n\theta\cos\theta - \cos n\theta \sin\theta.\end{aligned}$$

Adding the two equations and dividing both sides by $\sin\theta$, we get

$$U_n(x) + U_{n-2}(x) = 2xU_{n-1}(x), \qquad n \geq 1 \quad \textbf{[2 points]}.$$

This is equivalent to the recurrence in the problem.

(3) By the change of variables $x = \cos\theta$, $0 \le \theta \le \pi$, with $dx = -\sin\theta d\theta = -\sqrt{1-x^2}d\theta$,

$$\begin{aligned}
&\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{1/2}dx \\
&= \int_0^{\pi} \sin(m+1)\theta \sin(n+1)\theta d\theta \quad \textbf{[2 points]} \\
&= \frac{1}{2}\int_0^{\pi} (\cos(m-n)\theta + \cos(m+n)\theta)\, d\theta \quad \textbf{[2 points]} \\
&= \frac{\pi}{2}\delta_{m,n}.
\end{aligned}$$

(4) Since $\deg U_n(x) = 2^n$ for all $n \ge 0$, we have $\hat{U}_n(x) = 2^{-n}U_n(x)$. Dividing both sides of the recurrence in (2) by 2^{n+1}, we obtain $b_n = 0$ and $\lambda_n = 1/4$ **[2 points]**. □

Problem 1.4. Let $\{P_n(x)\}_{n\ge 0}$ be the monic OPS for a linear functional $\mathcal{L}$ with three-term recurrence

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x), \qquad n \ge 0.$$

(1) Prove that

$$P_n(x) = \begin{vmatrix} x-b_0 & 1 & & 0 \\ \lambda_1 & x-b_1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & \lambda_{n-1} & x-b_{n-1} \end{vmatrix}.$$

(2) Prove that

$$P_n(x) = \begin{vmatrix} x-b_0 & \sqrt{\lambda_1} & & 0 \\ \sqrt{\lambda_1} & x-b_1 & \ddots & \\ & \ddots & \ddots & \sqrt{\lambda_{n-1}} \\ 0 & & \sqrt{\lambda_{n-1}} & x-b_{n-1} \end{vmatrix}.$$

(3) Using (2) prove that if $b_n \in \mathbb{R}$ and $\lambda_n > 0$ for all, then $P_n(x)$ has real roots only.

Solution. (1) Let $Q_n(x)$ be the determinant on the right-hand side. Expanding the determinant along the last row, we obtain the recursion

$$Q_n(x) = (x-b_{n-1})Q_{n-1}(x) - \lambda_{n-1}Q_{n-2}(x) \quad \textbf{[2 points]}.$$

Since $P_n(x)$ and $Q_n(x)$ satisfy the same recurrence with with the initial conditions $Q_0(x) = 1$ and $Q_1(x) = x - b_0$, we obtain that $Q_n(x) = P_n(x)$.

(2) Let $A_n = (\alpha_{i,j})$ be the matrix in (1) and let B_n be the matrix in (2). Then it suffices to find an invertible diagonal matrix $D = \mathrm{diag}(d_i)$ such that $B_n = DA_nD^{-1}$ **[3 points]**. To do this, observe that $DA_nD^{-1} = (d_i\alpha_{i,j}d_j^{-1})$. Since B_n and DA_nD^{-1} are tri-diagonal matrices, we have $B_n = DA_nD^{-1}$ if and only if the following hold:

$$\beta_{i,i} = d_i\alpha_{i,i}d_i^{-1}, \tag{1.1}$$

$$\beta_{i,i+1} = d_i\alpha_{i,i+1}d_{i+1}^{-1}, \tag{1.2}$$

$$\beta_{i+1,i} = d_{i+1}\alpha_{i+1,i}d_i^{-1}. \tag{1.3}$$

Since $\alpha_{i,i+1} = 1$ and $\beta_{i,i+1} = \sqrt{\lambda_{i+1}}$, (1.2) is equivalent to $d_{i+1} = d_i/\sqrt{\lambda_{i+1}}$. Indeed, if we set $d_0 = 1$ and $d_{i+1} = d_i/\sqrt{\lambda_{i+1}}$, then all three conditions above hold **[3 points]**.

Note: Alternatively, it can be proved directly that the right-hand side of the equation satisfies the same recurrence as $P_n(x)$.

(3) Since the zeros of $P_n(x)$ are the eigenvalues of a real symmetric matrix, they are real **[2 points]**. □

2. Homework 2 (Due: Oct 5)

Problem 2.1. Let id be the identity permutation.

(1) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^2 = id$.
(2) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^3 = id$.
(3) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^4 = id$.
(4) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^5 = id$.
(5) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^6 = id$.

Solution. We have $\pi^k = id$ if and only if every cycle of π is of length divisible by k. For example, if $\pi^6 = id$, then the decreasing sequence of the lengths of cycles of π must be (6), $(3,3)$, $(3,2,1)$, $(3,1,1,1)$, $(2,2,2)$, $(2,2,1,1)$, $(2,1,1,1,1)$, $(1,1,1,1,1,1)$. The number of such permutations is $5!$, $\binom{6}{3}\frac{1}{2}2^2$, $\binom{6}{3}\binom{3}{2}\cdot 2$, $\binom{6}{3}\cdot 2$, $5\cdot 3$, $\binom{6}{4}\cdot 3$, $\binom{6}{2}$, 1, respectively. In this way we get the answers as follows.

(1) 76 **[2 points]**
(2) 81 **[2 points]**
(3) 256 **[2 points]**
(4) 145 **[2 points]**
(5) 396 **[2 points]**

□

Problem 2.2. Let $c_1, \dots, c_n$ be a sequence of nonnegative integers such that $\sum_{i=1}^n ic_i = n$. Show that the number of permutations $\pi \in \mathfrak{S}_n$ with c_i cycles of length i for all $i = 1, \dots, n$ is

$$\frac{n!}{\prod_{i=1}^n i^{c_i} c_i!}.$$

Solution. Let X be the set of such permutations. We construct a map $\phi : \mathfrak{S}_n \to X$ as follows. Given $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, let $\phi(\sigma)$ be the permutation whose cycle notation is obtained from the word $\sigma_1 \cdots \sigma_n$ by placing parentheses so that the first c_1 cycles are of length 1, the next c_2 cycles are of length 2, and so on **[3 points]**. By the construction, this gives a map $\phi : \mathfrak{S}_n \to X$.

For any $\pi \in X$, there are $c_i!$ ways to arrange its c_i cycles and i ways to cyclically shift each each of these cycles. Therefore, there are $\prod_{i=1}^n i^{c_i} c_i!$ permutations $\sigma \in \mathfrak{S}_n$ whose image under ϕ is π **[4 points]**. This shows that $|X| = |\mathfrak{S}_n| / \prod_{i=1}^n i^{c_i} c_i!$ as desired **[3 points]**. □

Problem 2.3. For $\pi \in \mathfrak{S}_n$, let $\ell(\pi)$ be the smallest number of simple transpositions whose product is π. Prove that $\ell(\pi) = \operatorname{inv}(\pi)$.

Solution. Suppose that $\pi = s_1 \cdots s_r$ for some simple transpositions s_i's. Since multipling a simple transposition increases or decreases the number of inversions by 1, we have $r \ge \operatorname{inv}(\pi)$ **[3 points]**. Hence $\ell(\pi) \ge \operatorname{inv}(\pi)$ **[2 points]**.

On the other hand, we can find an expression $\pi = s_1 \cdots s_r$ with $r = \operatorname{inv}(\pi)$ by sorting $\pi = \pi_1 \cdots \pi_n$ **[3 points]** because multiplying the simple transposition $(i, i+1)$ to the right of $\pi = \pi_1 \cdots \pi_n$ gives

$$\pi(i, i+1) = \pi_1 \cdots \pi_{i-1}\pi_{i+1}\pi_i\pi_{i+1} \cdots \pi_n.$$

This implies $\ell(\pi) \le \operatorname{inv}(\pi)$ **[2 points]**. Thus, $\ell(\pi) = \operatorname{inv}(\pi)$. □

Problem 2.4. Prove that

$$\sum_{\pi \in \mathfrak{S}_n} q^{\operatorname{inv}(\pi)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Solution. We proceed by induction on n. If $n = 1$, it is true. Let $n \ge 2$ and suppose the statement holds for $n-1$. Every $\pi \in \mathfrak{S}_n$ is obtained from $\sigma \in \mathfrak{S}_{n-1}$ by inserting n after j integers from the beginning for some $0 \le j \le n-1$ **[3 points]**. This construction gives $\operatorname{inv}(\pi) = \operatorname{inv}(\sigma) + j$

[3 points]. Thus

$$\sum_{\pi\in\mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = \sum_{\sigma\in\mathfrak{S}_{n-1}} \sum_{j=0}^{n-1} q^{\mathrm{inv}(\sigma)+j} = \sum_{\sigma\in\mathfrak{S}_{n-1}} q^{\mathrm{inv}(\sigma)}(1+q+\cdots+q^{n-1}) \quad \textbf{[2 points]}$$

$$= (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}) \quad \textbf{[2 points]}.$$

Thus the statement is also true for n and we are done. □

3. Homework 3 (Due: Oct 19)

Problem 3.1. Suppose that $\{P_n(x)\}_{n\geq 0}$ is a monic OPS for a linear functional $\mathcal{L}$ with $\mathcal{L}(1)=1$ given by $P_{-1}(x)=0$, $P_0(x)=1$, and for $n\geq 0$,

$$P_{n+1}(x)=(x-n)P_n(x)-nP_{n-1}(x).$$

Compute the following.

(1) $\mathcal{L}(x^3)$
(2) $\mathcal{L}(P_{10}(x)P_{10}(x))$
(3) $\mathcal{L}(x^3P_{10}(x)P_{12}(x))$

Solution. We can compute these quantities using

$$\mathcal{L}(x^nP_r(x)P_s)=\lambda_1\cdots\lambda_s\sum_{\pi\in\mathrm{Motz}((0,r)\to(n,s))}\mathrm{wt}(\pi).$$

(1) $\mathcal{L}(x^3)=1$ [**3 points**]
(2) $\mathcal{L}(P_{10}(x)P_{10}(x))=10!$ [**3 points**]
(3) $\mathcal{L}(x^3P_{10}(x)P_{12}(x))=33\cdot 12!$ [**4 points**]

□

Problem 3.2. A *left-to-right minimum* of a permutation $\pi=\pi_1\cdots\pi_n$ is a number π_i such that $\pi_i=\min\{\pi_1,\dots,\pi_i\}$. Let $\mathrm{LRmin}(\pi)$ denote the number of left-to-right minima in π. For example, if $\pi=6741352$, then the left-to-right minima are $6,4,1$, hence $\mathrm{LRmin}(\pi)=3$. Prove that

$$\sum_{\pi\in\mathfrak{S}_n}\alpha^{\mathrm{cycle}(\pi)}=\sum_{\pi\in\mathfrak{S}_n}\alpha^{\mathrm{LRmin}(\pi)}.$$

Solution. We can uniquely write the cycles of a permutation $\pi\in\mathfrak{S}_n$ so that each cycle starts with its smallest element and the cycles are listed in the decreasing order of their smallest elements [**3 points**]. For example,

$$\pi=(5,11)(3)(1,4,2,9,10,7,6,8).$$

Let $\widehat{\pi}$ be the permutation obtained from this list of cycles by deleting the parentheses [**4 points**]. In the example above,

$$\widehat{\pi}=5\,11\,3\,1\,4\,2\,9\,10\,7\,6\,8.$$

Then the first elements of the cycles of π are the left-to-right minima of $\widehat{\pi}$ [**3 points**]. Since $\pi\mapsto\widehat{\pi}$ is a bijection we have

$$\sum_{\pi\in\mathfrak{S}_n}\alpha^{\mathrm{cycle}(\pi)}=\sum_{\pi\in\mathfrak{S}_n}\alpha^{\mathrm{LRmin}(\pi)}.$$ □

Problem 3.3. Suppose that $\{P_n(x)\}_{n\geq 0}$ is a monic OPS given by $P_{-1}(x)=0$, $P_0(x)=1$, and for $n\geq 0$,

$$P_{n+1}(x)=(x-1)P_n(x)-nP_{n-1}(x).$$

Prove that μ_n is equal to the number of involutions in $\mathfrak{S}_n$. (An involution is a permutation π such that π^2 is the identity map.)

Solution. Recall the bijection $\phi:\mathrm{CH}_n\to\Pi_n$ between the Charlier histories of length n and the set partitions of $[n]$ for the case $b_n=n+1$ and $\lambda_n=n$ [**3 points**]. If $b_n=1$ and $\lambda_n=n$, then by restricting this map to the Charlier histories with 0 label for every horizontal step, the images are the set partitions in which every block is of size 1 or 2 [**4 points**]. Then we can identify such a set partition as an involution [**3 points**]. This implies the statement in the problem. □

Problem 3.4. Suppose that $\{P_n(x)\}_{n\geq 0}$ is a monic OPS given by $P_{-1}(x)=0$, $P_0(x)=1$, and for $n\geq 0$,

$$P_{n+1}(x)=(x-b_n)P_n(x)-\lambda_nP_{n-1}(x),$$

where $\lambda_n\neq 0$ for all $n\geq 1$.

Using the fact $\mu_n=\sum_{\pi\in\mathrm{Motz}_n}\mathrm{wt}(\pi)$, prove that $\mu_{2n+1}=0$ for all $n\geq 0$ if and only if $b_n=0$ for all $n\geq 0$.

Solution. ($\Leftarrow$): Suppose $b_n = 0$ for all $n \geq 0$. Then μ_n is the generating function for Dyck paths, hence $\mu_{2n+1} = 0$ [**3 points**].

($\Rightarrow$): Suppose that $\mu_{2n+1} = 0$ for all $n \geq 0$. Then we prove $b_n = 0$ for all $n \geq 0$ by induction on n. Since $\mu_1 = b_0$, we have $b_0 = 0$ [**3 points**]. Suppose that $b_i = 0$ for all $0 \leq i < n$. Then

$$\mu_{2n+1} = \sum_{\mathrm{Motz}_{2n+1}} \mathrm{wt}(\pi) = b_n \lambda_1 \cdots \lambda_n \quad [\textbf{4 points}]$$

because all Motzkin paths in Motz_{2n+1} except $U^n H D^n$ has weight 0. Since $\mu_{2n+1} = 0$ and $\lambda_i \leq 0$ for all i, we obtain $b_n = 0$. By induction, $b_n = 0$ for all $n \geq 0$. □

4. Homework 4 (Due: Nov 9)

Problem 4.1. Let G be the directed graph whose vertex set V and (directed) edge set E are given by

$$V = \{(i,j) : 0 \le i,j \le 5\},$$
$$E = \{(i,j) \to (i+1,j) : 0 \le i \le 4, 0 \le j \le 5\} \cup \{(i,j) \to (i,j+1) : 0 \le i \le 5, 0 \le j \le 4\}.$$

(1) Find the number of paths from $(0,0)$ to $(5,5)$.
(2) Find the number of paths from $(0,0)$ to $(5,5)$ that do not visit $(3,3)$.
(3) Find the number of paths from $(0,0)$ to $(5,5)$ that do not visit any of $(1,3),(3,3),(4,3)$. (Write you answer as a single determinant.)
(4) Let $\boldsymbol{A} = (A_1,A_2,A_3)$ and $\boldsymbol{B} = (B_1,B_2,B_3)$, where $A_1 = (0,0)$, $A_2 = (1,0)$, $A_3 = (2,0)$, $B_1 = (5,5)$, $B_2 = (5,4)$, and $B_3 = (5,3)$. Find the cardinality of the set $\mathrm{NI}(\boldsymbol{A} \to \boldsymbol{B})$. (Write you answer as a single determinant.)

Solution. (1) $\binom{10}{5}$ **[2 points]**

(2) $\binom{10}{5} - \binom{6}{2}\binom{4}{2}$ **[2 points]**

(3) Let $A_1 = (0,0), B_1 = (5,5)$ and $A_2 = B_2 = (1,3), A_3 = B_3 = (3,3), A_4 = B_4 = (4,3)$ **[2 points]**. Then by the LGV lemma, the answer is

$$\det \begin{pmatrix} \binom{10}{5} & \binom{4}{1} & \binom{6}{3} & \binom{7}{3} \\ \binom{6}{2} & 1 & \binom{2}{0} & \binom{3}{0} \\ \binom{4}{2} & 0 & 1 & \binom{1}{0} \\ \binom{3}{2} & 0 & 0 & 1 \end{pmatrix}. \quad \textbf{[2 points]}$$

(4) By the LGV lemma, the answer is

$$\det \begin{pmatrix} \binom{10}{5} & \binom{9}{4} & \binom{8}{3} \\ \binom{9}{4} & \binom{8}{4} & \binom{7}{3} \\ \binom{8}{3} & \binom{7}{3} & \binom{6}{3} \end{pmatrix}. \quad \textbf{[2 points]}$$

□

Problem 4.2. Evaluate the determinants. Here, C_n is the nth Catalan number.

(1) $\det\left(C_{i+j}\right)_{i,j=0}^{2023}$
(2) $\det\left(\binom{2i+2j}{i+j}\right)_{i,j=0}^{2023}$

Solution. (1) Consider the directed graph $G = (V,E)$ where $V = \mathbb{Z} \times \mathbb{Z}_{\ge 0}$ and edges consisting of up steps $U = (1,1)$ and down steps $D = (1,-1)$. Let $\boldsymbol{A} = (A_0, A_1, \dots, A_{2023})$ and $\boldsymbol{B} = (B_0, B_1, \dots, B_{2023})$, where $A_i = (-i, 0)$ and $B_i = (i, 0)$ **[2 points]**. Then

$$\det\left(C_{i+j}\right)_{i,j=0}^{2023} = |\,\mathrm{NI}(\boldsymbol{A} \to \boldsymbol{B})|. \quad \textbf{[2 points]}$$

Since there is only one element in $\mathrm{NI}(\boldsymbol{A} \to \boldsymbol{B})$, the answer is 1.

(2) We have shown that

$$\binom{2n}{n} = \sum_{\pi \in \mathrm{Dyck}_n} 2^{a(\pi)},$$

where $a(\pi)$ is the number of down steps in π touching the x-axis **[2 points]**. Define an edge-weight $w : E \to K$ for the graph G above by $w(U) = 1$ and $w(D) = 2$ if D has starting height 1 and $w(D) = 1$ otherwise. Then

$$\det\left(\binom{2i+2j}{i+j}\right)_{i,j=0}^{2023} = \sum_{\boldsymbol{p} \in \mathrm{NI}(\boldsymbol{A} \to \boldsymbol{B})} \mathrm{sgn}(\boldsymbol{p}) w(p). \quad \textbf{[2 points]}$$

Since there is only one element in $\mathrm{NI}(\boldsymbol{A} \to \boldsymbol{B})$, which has 2023 down steps with starting height 1 in total, the answer is 2^{2023} **[2 points]**. □

Problem 4.3. Let $\{P_n(x)\}_{n\ge 0}$ be a monic OPS satisfying

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x),$$

and let μ_n be the nth moment. Suppose that $\lambda_n > 0$ for all $n \ge 1$ and $b_n \ge 0$ for all $n \ge 0$. Prove or disprove each statement.

(1) For all $n \ge 0$,
$$\det(\mu_{i+j})_{i,j=0}^n > 0.$$
(2) For all $n \ge 0$,
$$\det(\mu_{2i+2j})_{i,j=0}^n > 0.$$
(3) If $b_k = 0$ for all $k \ge 0$, then for all $n \ge 0$,
$$\det(\mu_{2i+2j})_{i,j=0}^n > 0.$$
(4) Let $\{r_n\}_{n\ge 0}$ and $\{s_n\}_{n\ge 0}$ be strictly increasing sequences of nonnegative even integers. If $b_k = 0$ for all $k \ge 0$, then for all $n \ge 0$,
$$\det(\mu_{r_i+s_j})_{i,j=0}^n > 0.$$

Solution. (1) We have

$$\det(\mu_{i+j})_{i,j=0}^n = \lambda_1^n \lambda_2^{n-1} \cdots \lambda_n^1 > 0. \quad \textbf{[2 points]}$$

(2) Since there are figure "X"'s, we cannot determine the positivity of this determinant using the LGV lemma **[2 points]**. (I do not know if this is positive or not. Please let me know if you have a proof or a counterexample.)

(3) We have

$$\det(\mu_{2i+2j})_{i,j=0}^n = \Delta_n(2) = (\lambda_1\lambda_2)^n(\lambda_3\lambda_4)^{n-1}\cdots(\lambda_{2n-1}\lambda_{2n})^1 > 0. \quad \textbf{[2 points]}$$

(4) Let $\boldsymbol{A} = (A_0, A_1, \dots, A_n)$ and $\boldsymbol{B} = (B_0, B_1, \dots, B_n)$, where $A_i = (-r_i, 0)$ and $B_i = (s_i, 0)$. Then

$$\det(\mu_{r_i+s_j})_{i,j=0}^n = \sum_{\boldsymbol{p}\in \mathrm{NI}(\boldsymbol{A}\to\boldsymbol{B})} \mathrm{sgn}(\boldsymbol{p}) w(\boldsymbol{p}),$$

where $\boldsymbol{p} = (p_0, \dots, p_n)$ is a nonintersecting n-path such that each p_i is a Dyck path. Since r_i and s_j are even, for every point (a,b) of a path p_i, $a+b$ is even **[2 points]**. This means that there is no figure "X" among p_i's. Therefore, each p_i is a path from A_i to B_i and $\mathrm{sgn}(\boldsymbol{p}) = 1$. Hence,

$$\det(\mu_{r_i+s_j})_{i,j=0}^n = \sum_{\boldsymbol{p}\in \mathrm{NI}(\boldsymbol{A}\to\boldsymbol{B})} \mathrm{wt}(\boldsymbol{p}) > 0. \quad \textbf{[2 points]}$$ □

Problem 4.4. Prove the following two q-binomial theorems:

$$(1+x)(1+qx)\cdots(1+q^{n-1}x) = \sum_{k=0}^n q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k,$$

$$\frac{1}{(1-x)(1-qx)\cdots(1-q^{n-1}x)} = \sum_{k=0}^\infty \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k.$$

Solution. We have

$$(1+x)(1+qx)\cdots(1+q^{n-1}x) = \sum_{k=0}^n \sum_{0\le i_1<\cdots<i_k\le n-1} q^{i_1+\cdots+i_k} x^k. \quad \textbf{[2 points]}$$

Replacing $(i_1, \dots, i_k)$ by $(j_1, \dots, j_k)$ with $j_t = i_t - t + 1$, we get

$$(1+x)(1+qx)\cdots(1+q^{n-1}x) = \sum_{k=0}^n q^{\binom{k}{2}} \sum_{0\le j_1\le\cdots\le j_k\le n-k} q^{j_1+\cdots+j_k} x^k. \quad \textbf{[2 points]}$$

Since

$$\sum_{0\le j_1\le\cdots\le j_k\le n-k} q^{j_1+\cdots+j_k} = \sum_{\lambda\subseteq((n-k)^k)} q^{|\lambda|} = \begin{bmatrix} n \\ k \end{bmatrix}_q, \quad \textbf{[2 points]}$$

we obtain the first identity.

For the second identity, note that

$$\frac{1}{(1-x)(1-qx)\cdots(1-q^{n-1}x)} = \sum_{k=0}^{\infty} \sum_{0\le i_1\le\cdots\le i_k\le n-1} q^{i_1+\cdots+i_k} x^k. \quad \textbf{[2 points]}$$

Since

$$\sum_{0\le i_1\le\cdots\le i_k\le n-1} q^{i_1+\cdots+i_k} = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q, \quad \textbf{[2 points]}$$

we obtain the second identity. □