HOMEWORK

Contents

1. Homework 1 (Due: Sep 21)

Problem 1.1. Let $\mathcal{L}$ be a positive-definite linear functional with monic OPS $\{P_n(x)\}_{n\geq 0}$. Prove the following extremal property: for any monic real polynomial $\pi(x) \neq P_n(x)$ of degree n,

$$\mathcal{L}(P_n(x)^2) < \mathcal{L}(\pi(x)^2).$$

Problem 1.2. Let $\mathcal{L}$ be a linear functional such that $\Delta_n \neq 0$ for all $n \geq 0$. Prove that if $\pi(x)$ is a polynomial such that $\mathcal{L}(x^k \pi(x)) = 0$ for all $k \geq 0$, then $\pi(x) = 0$.

Problem 1.3. The *Tchebyshev polynomials of the second kind* $U_n(x)$ are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta, \qquad n \geq 0.$$

(1) Prove that $U_n(x)$ is a polynomial of degree n.
(2) Prove that
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \qquad n \geq 0,$$
where $U_{-1}(x) = 0$ and $U_0(x) = 1$.
(3) Prove that
$$\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{1/2}dx = \frac{\pi}{2}\delta_{m,n}.$$
(4) Find the 3-term recurrence for the normalized Tchebyshev polynomials of the second kind. More precisely, find the numbers b_n and λ_n such that
$$\hat{U}_{n+1}(x) = (x-b_n)\hat{U}_n(x) - \lambda_n \hat{U}_{n-1}(x), \qquad n \geq 0,$$
where $\hat{U}_n(x)$ is the monic polynomial that is a scalar multiple of $U_n(x)$.

Problem 1.4. Let $\{P_n(x)\}_{n\geq 0}$ be the monic OPS for a linear functional $\mathcal{L}$ with three-term recurrence

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x), \qquad n \geq 0.$$

(1) Prove that
$$P_n(x) = \begin{vmatrix} x-b_0 & 1 & & 0 \\ \lambda_1 & x-b_1 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & \lambda_{n-1} & x-b_{n-1} \end{vmatrix}.$$
(2) Prove that
$$P_n(x) = \begin{vmatrix} x-b_0 & \sqrt{\lambda_1} & & 0 \\ \sqrt{\lambda_1} & x-b_1 & \ddots & \\ & \ddots & \ddots & \sqrt{\lambda_{n-1}} \\ 0 & & \sqrt{\lambda_{n-1}} & x-b_{n-1} \end{vmatrix}.$$
(3) Using (2) prove that if $b_n \in \mathbb{R}$ and $\lambda_n > 0$ for all, then $P_n(x)$ has real roots only.

2. Homework 2 (Due: Oct 5)

Problem 2.1. Let id be the identity permutation.

(1) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^2 = id$.
(2) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^3 = id$.
(3) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^4 = id$.
(4) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^5 = id$.
(5) Find the number of permutations $\pi \in \mathfrak{S}_6$ such that $\pi^6 = id$.

Problem 2.2. Let $c_1, \dots, c_n$ be a sequence of nonnegative integers such that $\sum_{i=1}^n ic_i = n$. Show that the number of permutations $\pi \in \mathfrak{S}_n$ with c_i cycles of length i for all $i = 1, \dots, n$ is

$$\frac{n!}{\prod_{i=1}^n i^{c_i} c_i!}.$$

Problem 2.3. For $\pi \in \mathfrak{S}_n$, let $\ell(\pi)$ be the smallest number of simple transpositions whose product is π. Prove that $\ell(\pi) = \mathrm{inv}(\pi)$.

Problem 2.4. Prove that

$$\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

3. Homework 3 (Due: Oct 19)

Problem 3.1. Suppose that $\{P_n(x)\}_{n\geq 0}$ is a monic OPS for a linear functional $\mathcal{L}$ with $\mathcal{L}(1)=1$ given by $P_{-1}(x)=0$, $P_0(x)=1$, and for $n\geq 0$,

$$P_{n+1}(x)=(x-n)P_n(x)-nP_{n-1}(x).$$

Compute the following.

(1) $\mathcal{L}(x^3)$
(2) $\mathcal{L}(P_{10}(x)P_{10}(x))$
(3) $\mathcal{L}(x^3P_{10}(x)P_{12}(x))$

Problem 3.2. A *left-to-right minimum* of a permutation $\pi=\pi_1\cdots\pi_n$ is a number π_i such that $\pi_i=\min\{\pi_1,\dots,\pi_i\}$. Let $\mathrm{LRmin}(\pi)$ denote the number of left-to-right minima in π. For example, if $\pi=6741352$, then the left-to-right minima are $6,4,1$, hence $\mathrm{LRmin}(\pi)=3$. Prove that

$$\sum_{\pi\in\mathfrak{S}_n}\alpha^{\mathrm{cycle}(\pi)}=\sum_{\pi\in\mathfrak{S}_n}\alpha^{\mathrm{LRmin}(\pi)}.$$

Problem 3.3. Suppose that $\{P_n(x)\}_{n\geq 0}$ is a monic OPS given by $P_{-1}(x)=0$, $P_0(x)=1$, and for $n\geq 0$,

$$P_{n+1}(x)=(x-1)P_n(x)-nP_{n-1}(x).$$

Prove that μ_n is equal to the number of involutions in $\mathfrak{S}_n$. (An involution is a permutation π such that π^2 is the identity map.)

Problem 3.4. Suppose that $\{P_n(x)\}_{n\geq 0}$ is a monic OPS given by $P_{-1}(x)=0$, $P_0(x)=1$, and for $n\geq 0$,

$$P_{n+1}(x)=(x-b_n)P_n(x)-\lambda_nP_{n-1}(x),$$

where $\lambda_n\neq 0$ for all $n\geq 1$.

Using the fact $\mu_n=\sum_{\pi\in\mathrm{Motz}_n}\mathrm{wt}(\pi)$, prove that $\mu_{2n+1}=0$ for all $n\geq 0$ if and only if $b_n=0$ for all $n\geq 0$.

4. Homework 4 (Due: Nov 9)

Problem 4.1. Let G be the directed graph whose vertex set V and (directed) edge set E are given by

$$V = \{(i,j) : 0 \le i,j \le 5\},$$
$$E = \{(i,j) \to (i+1,j) : 0 \le i \le 4, 0 \le j \le 5\} \cup \{(i,j) \to (i,j+1) : 0 \le i \le 5, 0 \le j \le 4\}.$$

(1) Find the number of paths from $(0,0)$ to $(5,5)$.
(2) Find the number of paths from $(0,0)$ to $(5,5)$ that do not visit $(3,3)$.
(3) Find the number of paths from $(0,0)$ to $(5,5)$ that do not visit any of $(1,3),(3,3),(4,3)$. (Write you answer as a single determinant.)
(4) Let $\boldsymbol{A} = (A_1, A_2, A_3)$ and $\boldsymbol{B} = (B_1, B_2, B_3)$, where $A_1 = (0,0)$, $A_2 = (1,0)$, $A_3 = (2,0)$, $B_1 = (5,5)$, $B_2 = (5,4)$, and $B_3 = (5,3)$. Find the cardinality of the set $\mathrm{NI}(\boldsymbol{A} \to \boldsymbol{B})$. (Write you answer as a single determinant.)

Problem 4.2. Evaluate the determinants. Here, C_n is the nth Catalan number.

(1) $\det \left(C_{i+j}\right)_{i,j=0}^{2023}$
(2) $\det \left(\binom{2i+2j}{i+j}\right)_{i,j=0}^{2023}$

Problem 4.3. Let $\{P_n(x)\}_{n\ge0}$ be a monic OPS satisfying

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_n P_{n-1}(x),$$

and let μ_n be the nth moment. Suppose that $\lambda_n > 0$ for all $n \ge 1$ and $b_n \ge 0$ for all $n \ge 0$. Prove or disprove each statement.

(1) For all $n \ge 0$,
$$\det(\mu_{i+j})_{i,j=0}^n > 0.$$
(2) For all $n \ge 0$,
$$\det(\mu_{2i+2j})_{i,j=0}^n > 0.$$
(3) If $b_k = 0$ for all $k \ge 0$, then for all $n \ge 0$,
$$\det(\mu_{2i+2j})_{i,j=0}^n > 0.$$
(4) Let $\{r_n\}_{n\ge0}$ and $\{s_n\}_{n\ge0}$ be strictly increasing sequences of nonnegative even integers. If $b_k = 0$ for all $k \ge 0$, then for all $n \ge 0$,
$$\det(\mu_{r_i+s_j})_{i,j=0}^n > 0.$$

Problem 4.4. Prove the following two q-binomial theorems:

$$(1+x)(1+qx)\cdots(1+q^{n-1}x) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k,$$
$$\frac{1}{(1-x)(1-qx)\cdots(1-q^{n-1}x)} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k.$$