Introduction to orthogonal polynomials.

Prerequisites: Calculus, Linear Algebra
Discrete Math.

In calculus, we know

$$2\cos m\theta \cos n\theta = \cos(m+n)\theta + \cos(m-n)\theta$$

(m, n: nonnegative integers).

$$\int_0^\pi \cos m\theta \cos n\theta \, d\theta = 0 \quad (\text{if } m \neq n)$$

$\Rightarrow$ $\cos m\theta$, $\cos n\theta$ are orthogonal over $(0,\pi)$.

We knows $\cos n\theta$ is a polynomial in $\cos\theta$.

e.g. $n=1$. $\cos n\theta = \cos\theta \rightarrow T_1(x) = x$

$n=2$, $\cos 2\theta = 2\cos^2\theta - 1 \rightarrow T_2(x) = 2x^2-1$.

$\vdots$

$\Rightarrow \cos n\theta = T_n(\cos\theta)$

$T_0(x) = 1$.

$T_1(x) = x$

$T_2(x) = 2x^2-1$

$T_3(x) = 4x^3-3x$

$\vdots$

$$\int_0^{\pi} \cos m\theta \cos n\theta \, d\theta = 0 \quad (\text{if } m \neq n)$$

let $x = \cos\theta$.

$$\int_{-1}^{1} T_m(x) T_n(x) \frac{1}{\sqrt{1-x^2}} dx = 0 \quad (m \neq n)$$

$\Rightarrow$ $\{T_n(x)\}_{n \geq 0}$ is orthogonal with respect to weight function $\frac{1}{\sqrt{1-x^2}}$.

$T_n(x)$: Tchebyshev polynomials.

Recall V: Inner prod space with Inner prod $\langle , \rangle$.

$u_1, u_2, \ldots, u_n \in V$ are orthogonal if $\langle u_i, u_j \rangle = 0$ if $i \neq j$.

$V = \mathbb{R}[x]$ = space of polynomials with real coeffs.

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x) g(x) \frac{1}{\sqrt{1-x^2}} dx.$$

Def) Suppose $w(x)$ is nonnegative and integrable on (a,b) with $\int_a^b w(x)\,dx > 0$, and $\int_a^b x^n w(x)\,dx < \infty$ for all $n \geq 0$.

A sequence of polynomials $\{P_n(x)\}_{n\geq 0}$ is an <u>orthogonal polynomial sequence</u> (OPS) w.r.t. <u>weight function</u> $w(x)$ (also called <u>measure</u>) if following hold:

① $\deg P_n(x) = n$ for all $n \geq 0$.

② $\int_a^b P_n(x) P_m(x) w(x)\,dx = 0$ if $m \neq n$.

Note: $\int_a^b f(x) w(x)\,dx \geq 0$ if $f(x)$ poly, $f(x) \geq 0$.

There is another way to study OPS.

For a poly $f(x)$, let

$$\mathcal{L}(f(x)) = \int_a^b f(x) w(x)\,dx.$$

Then $\mathcal{L}(f(x))$ is completely determined by the <u>moments</u> $\mu_n = \int_a^b x^n w(x)\,dx$.

eg. $$\int_a^b (x^2+2x) w(x)\,dx = \int_a^b x^2 w(x)\,dx + 2\int_a^b x\, w(x)\,dx = \mu_2 + 2\mu_1$$

$\mu_0, \mu_1, \mu_2, \ldots$: moment seq

Def) Let $\mathcal{L}$ be a linear functional defined on the space of polynomials in x.

$(\mathbb{R}[x], \mathbb{C}[x].)$

$\{P_n(x)\}_{n\geq 0}$ is an OPS w.r.t. $\mathcal{L}$ if

① $\deg P_n(x) = n$, $n \geq 0$.

② $\mathcal{L}(P_n(x)^2) \neq 0$, $n \geq 0$

③ $\mathcal{L}(P_m(x) P_n(x)) = 0$, $m \neq n$.

Remark The moments of Tchebyshev poly

$$\mu_n = \int_{-1}^{1} x^n \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} \frac{\pi}{2^{2n}} \binom{2n}{n} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

ex) (Charlier poly).

The Charlier polynomials $P_n(x)$ are defined by

$$P_n(x) = \sum_{k=0}^{n} \binom{x}{k} \frac{(-a)^{n-k}}{(n-k)!}$$

$$_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$ binomial coeff.

$=$ # ways to select k elts from $\{1, \dots, n\}$.

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$$: poly in x deg k.

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

$$P_n(x) = \sum_{k=0}^{n} \binom{x}{k} \frac{(-a)^{n-k}}{(n-k)!}$$

The generating function for $P_n(x)$ is

$$G(x,w) = \sum_{n\geq 0} P_n(x)\, w^n$$

$$= \sum_{n\geq 0} \sum_{k=0}^{n} \binom{x}{k} \frac{(-a)^{n-k}}{(n-k)!} w^n \qquad \begin{matrix} k \to n \\ n-k \to m \end{matrix}$$

$$= \sum_{n\geq 0} \binom{x}{n} w^n \sum_{n\geq 0} \frac{(-a)^m}{m!} w^m$$

$$= (1+w)^x e^{-aw}$$

$$\Rightarrow \quad a^x G(x,v) G(x,w) = e^{-a(v+w)} \big(a(1+v)(1+w)\big)^x$$

$$\Rightarrow \quad \sum_{k\geq 0} \frac{a^k G(k,v)\, G(k,w)}{k!}$$

$$= \sum_{k\geq 0} \frac{e^{-a(v+w)} \big(a(1+v)(1+w)\big)^k}{k!}$$

$$= e^{-a(v+w)} \cdot e^{a(1+v)(1+w)} = e^{a(1+vw)}$$

$$= e^a \cdot e^{a(vw)} = e^a \sum_{k\geq 0} \frac{(avw)^k}{k!}$$

$$\sum_{k\geq 0} \frac{a^k G(k,v) G(k,w)}{k!} = e^a \sum_{k\geq 0} \frac{(avw)^k}{k!}$$

On the other hand,

$$LHS = \sum_{k\geq 0} \frac{a^k}{k!} \sum_{m,n\geq 0} p_m(k) p_n(k) v^m w^n$$

$$= \sum_{m,n\geq 0} \left(\sum_{k\geq 0} p_m(k) p_n(k) \frac{a^k}{k!} \right) v^m w^n$$

$$\Rightarrow \sum_{k\geq 0} p_m(k) p_n(k) \frac{a^k}{k!} = \begin{cases} 0 & \text{if } m\neq n \\ \dfrac{e^a a^n}{n!} & \text{if } m=n. \end{cases}$$

If we define a linear functional $\mathcal{L}$ by $\mathcal{L}(f(x)) = \sum_{k\geq 0} f(k) \frac{a^k}{k!}$

$\Rightarrow \{p_n(x)\}$: OPS w.r.t. $\mathcal{L}$.

<u>Note</u> It's possible to express $\mathcal{L}$ using integrals.

$$\mathcal{L}(f(x)) = \int_{-\infty}^{\infty} f(x)\, d\psi(x) \quad \text{(Riemann-Stieltjes)}$$

$\psi(x)$: step function with jump of magnitute $a^k/k!$ for each $k=0,1,2,\dots$