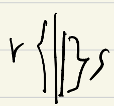
Thm $\left(\mu_{n,r,s} = \dfrac{\mathcal{L}(x^n P_r(x) P_s(x))}{\mathcal{L}(P_s(x)^2)}\right)$

$$\mu_{n,r,s} = \sum_{\pi \in Mot((0,r)\to(n,s))} wt(\pi)$$

pf) Induction n (r, s : arbitrary).

If $n=0$, $\mu_{0,r,s} = \dfrac{\mathcal{L}(P_r P_s)}{\mathcal{L}(P_s^2)} = \delta_{r,s}$,

and $Mot((0,r)\to(0,s)) = \begin{cases} \{\phi\}, & r=s \\ \phi, & r\neq s. \end{cases}$



$\Rightarrow$ RHS $= \delta_{r,s}$

Let $n \geq 1$. Suppose thm holds for $n-1$.

$$xP_r = P_{r+1} + b_r P_r + \lambda_r P_{r-1}$$

$$\mu_{n,r,s} = \frac{\mathcal{L}(x^n P_r P_s)}{\mathcal{L}(P_s^2)} = \frac{\mathcal{L}(x^{n-1}(xP_r)P_s)}{\mathcal{L}(P_s^2)}$$

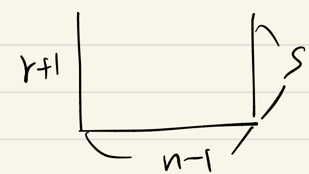
$$= \mathcal{L}(x^{n-1}(P_{r+1} + b_r P_r + \lambda_r P_{r-1})P_s) \,/\, \mathcal{L}(P_s^2)$$

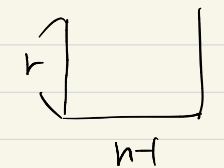
$$= \frac{\mathcal{L}(x^{n-1}P_{r+1}P_s)}{\mathcal{L}(P_s^2)} + b_r \frac{\mathcal{L}(x^{n-1}P_rP_s)}{\mathcal{L}(P_s^2)} + \lambda_r \frac{\mathcal{L}(x^{n-1}P_{r-1}P_s)}{\mathcal{L}(P_s^2)}$$

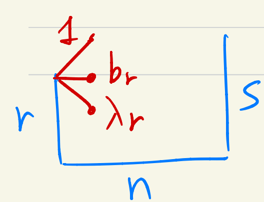
$$= \mu_{n-1,r+1,s} + b_r \mu_{n-1,r,s} + \lambda_r \mu_{n-1,r-1,s}$$

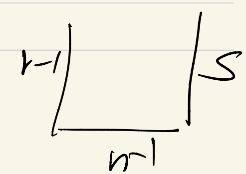
$$= \underbrace{\sum_{\pi \in Mot((0,r+1)\to(n-1,s))} wt(\pi)} + b_r \bigcirc + \underbrace{\lambda_r \bigcirc}$$

$$= \sum_{\pi \in Mot((0,r)\to(n,s))} wt(\pi)$$









$\square$

Cor $\mathcal{L}(p_n(x)^2) = \lambda_1 \cdots \lambda_n$.

Pf) Since $p_n(x) = x^n + Q(x)$, $(\deg Q \le n-1)$

← only Motzkin path

$$\mathcal{L}(p_n^2) = \mathcal{L}((x^n + Q)p_n)$$

$$= \mathcal{L}(x^n p_n) + \mathcal{L}(Q p_n)$$

$$= \mathcal{L}(x^n p_n)$$

$$= \frac{\mathcal{L}(x^n p_n p_0^2)}{\mathcal{L}(p_0^2)}$$

$$= \mu_{n,n,0}$$

$$= \sum_{\pi \in \mathrm{Mot}((0,n),(n,0))} \mathrm{wt}(\pi). = \lambda_1 \cdots \lambda_n.$$

□

§4.4. A bijective proof of Favard's thm.

Recall Favard's thm says

if $\{p_n(x)\}_{n\geq 0}$ satisfies

$$p_{n+1} = (x-b_n)p_n - \lambda_n p_{n-1}.$$

$\Rightarrow$ $\{p_n(x)\}$ is OPS for some $\mathcal{L}$.

To prove this thm bijectively we need to first construct $\mathcal{L}$.

Define
$$\mathcal{L}(x^n) := \sum_{\pi \in \mathrm{Mot}_n} wt(\pi).$$

Goal: Prove

$$\mathcal{L}(p_r(x)p_s(x)) = \lambda_1 \cdots \lambda_r \, \delta_{r,s}$$

bijectively!

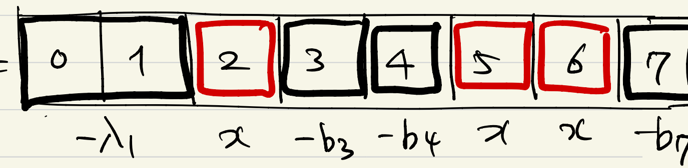
More generally, we will prove

Thm
$$\mathcal{L}(x^n p_r p_s) = \lambda_1 \cdots \lambda_s \sum_{\pi \in \mathrm{Mot}_{n,r,s}} wt(\pi)$$

$$\mathrm{Motz}_{n,r,s} = \mathrm{Motz}((0,r) \to (n,s))$$

Recall
$$p_n(x) = \sum_{T \in FT_n} wt(T)$$

T =



Let $wt'(T) = wt(T) / x^{(\#\text{ red mono})}$

$$\Rightarrow \quad p_n(x) = \sum_{T \in FT_n} wt'(T)\, x^{(\#\text{ red mono})}$$

$$\mathcal{L}(x^n p_r p_s) = \sum_{(T_1,T_2,\pi)\in X} wt'(T_1)\, wt'(T_2)\, wt(\pi)$$

X = set of triples (T_1, T_2, π) s.t.

for some $0 \le i \le r$, $0 \le j \le s$

① $T_1 \in FT_r$ with i red mono.

② $T_2 \in FT_s$ " j "

③ $\pi \in Mot_{n+i+j}$.

$$\left(\because\ \mathcal{L}\Big(x^n \sum_{T_1\in FT_r} wt'(T_1)\, x^i \sum_{T_2 \in FT_s} wt'(T_2)\, x^j\Big)\right.$$

$$\left. = \sum_{T_1\in FT_r} \sum_{T_2\in FT_s} \underbrace{\mathcal{L}(x^{n+i+j})}_{\substack{\| \\ \sum_{\pi\in Mot_{n+i+j}} wt(\pi)}}\right)$$

It suffices to show

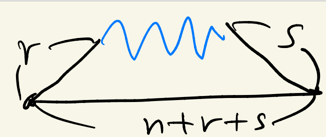
Thm

$$\sum_{(T_1,T_2,\pi)\in X} wt'(T_1)\, wt'(T_2)\, wt(\pi) = \lambda_1 \cdots \lambda_s \sum_{\pi \in Mot_{n,r,s}} wt(\pi)$$

Idea of Pf

Find a sign-reversing weight-preserving involution on X with fixed point set $\{(\phi, \phi, \pi) \mid \pi \in Y\}$.

Y = set of Motzkin paths $(0,0) \to (n,r,s)$



Here, a sign-reversing weight-preserving involution

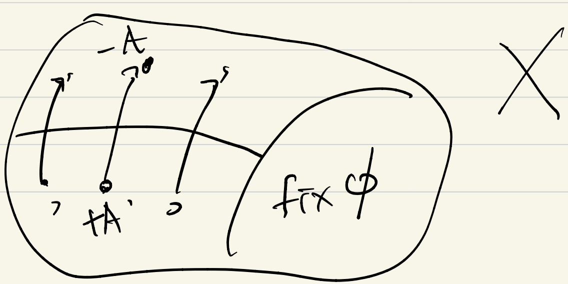
means $\phi : X \to X$

s.t. if $\phi(T_1, T_2, \pi) = (T_1', T_2', \pi')$

then $wt'(T_1') \, wt'(T_2') \, wt(\pi')$

$= - wt'(T_1) \, wt(T_2) \, wt(\pi)$

unless $(T_1, T_2, \pi) = (T_1', T_2', \pi')$
fixed point



Such a map ϕ implies

$$\sum_{A \in X} wt(A) = \sum_{A \in \mathrm{Fix}\,\phi} wt(A).$$

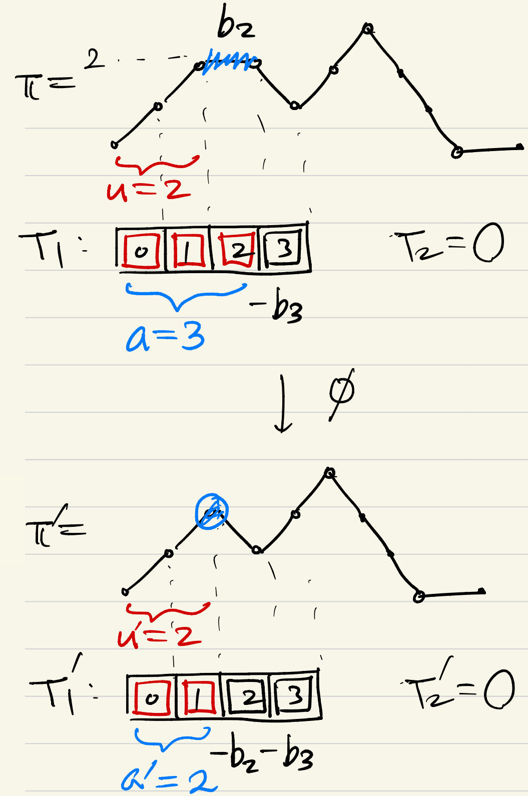
Let's find a s.r.w.p. inv $\phi : X \to X$.

Let $(T_1, T_2, \pi) \in X$.

$\pi = s_1 s_2 \cdots s_m$ (seq of steps).

u = max # up steps at beginning

a = max # red mono in T_1 at beginning



case 1-1

case 2-1

<u>case 1</u> $u < a$ $\qquad T_2' = T_2$.

<u>case 1-1</u> s_{u+1} is horizontal

collapse s_{u+1} into a point

change $(u+1)$st red mono to a black " .

$wt'(T_1') = wt'(T_1)(-b_u)$ $\Rightarrow$ $wt'(T_1')\, wt'(T_2')\, wt(\pi')$

$wt(\pi') = wt(\pi) / b_u$ $\qquad = -wt'(T_1)\, wt'(T_2)\, wt(\pi)$

case 1-2 S_{u+1} is down.

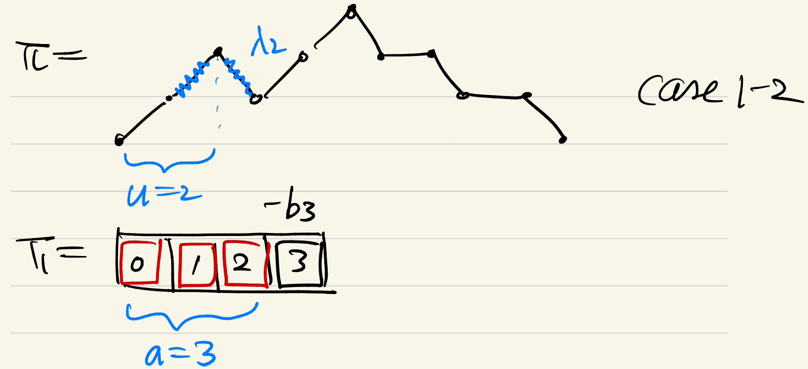
Collapse the "peak" $S_u S_{u+1}$ to ∘

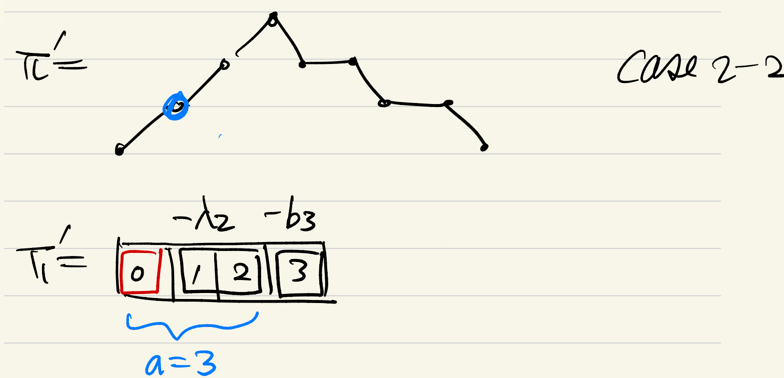
Change two red mono (uth and (u+1)st) to black domino

$$wt'(T_1') = wt'(T_1)(-\lambda_u)$$

$$wt(\pi') = wt(\pi)/\lambda_u$$

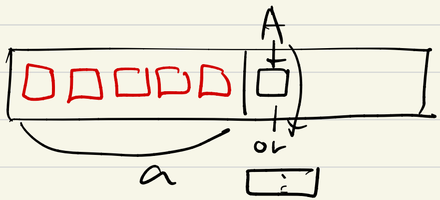
So we still have s.r.w.p.





Case 2 $u \geq a \neq r$

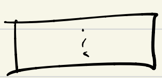
Let A be the $(a+1)$st tile in T_1

$T_1 =$ 

Case 2-1

$A =$ black mono. $= \square$

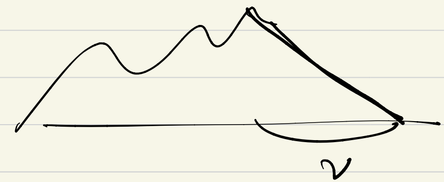
Case 2-2

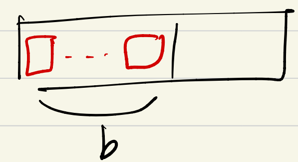
$A =$ 

Now remaining objects.

$u \geq a = r$

Case 3 $\longleftrightarrow$ Case 4.



$T_2 =$ 

Do similar things as in Case 1, 2

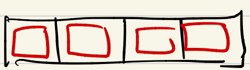
Case 5

$u \geq a = r, \quad v \geq b = s$

$\pi =$

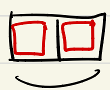
$u = 4 \qquad v = 3$

$T_1 =$



$a = 4 = r$

$T_2 =$



$b = 2 = s$

$\Rightarrow \phi(T_1, T_2, \pi) = (T_1, T_2, \pi).$

Y

Q.E.D.