Ch 5. Moments of classical orthogonal polynomials.

Note There are several ways to define OPS.

① coeffs $a_{n,k}$ of $P_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$

② generating function

$$\sum_{n\geq 0} P_n(x) t^n \quad \text{or} \quad \sum_{n\geq 0} P_n(x) \frac{t^n}{n!}$$

(ordinary) (exponential)

③ moments μ_n.

④ 3-term rec. coeffs b_n, λ_n

$$\left(P_{n+1} = (x-b_n) P_n - \lambda_n P_{n-1} \right)$$

§5.1. Tchebyshev polynomials.

Recall The Tchebyshev poly of 2nd kind are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta, \quad n \geq 0.$$

$$U_{n+1}(x) = 2x U_n(x) - U_{n-1}(x). \quad n \geq 0$$

$$U_{-1}(x) = 0, \quad U_0(x) = 1.$$

We know from calculus

$$\int_{-1}^{1} U_m(x) U_n(x) (1-x^2)^{1/2} dx = \frac{\pi}{2} \delta_{m,n}.$$

$\{U_n(x)\}_{n\geq 0}$ is an OPS for $\mathcal{L}$ where

$$\mathcal{L}(f(x)) = \frac{2}{\pi} \int_{-1}^{1} f(x) (1-x^2)^{1/2} dx.$$

$$(\mathcal{L}(1) = 1).$$

$U_n(x)$ has coeff 2^n, $n \geq 0$.

monic Tchebyshev

$\hat{U}_n(x) = 2^{-n} U_n(x) \quad (n \geq 0).$

If we divide 3-r.r. for U_n by 2^{n+1}

$$\Rightarrow \hat{U}_{n+1}(x) = x\hat{U}_n(x) - \frac{1}{4}\hat{U}_{n-1}(x) \quad (n \geq 0).$$

$b_n = 0$, $\lambda_n = \frac{1}{4}$.

$\{\hat{U}_n(x)\}$ is OPS for $\mathcal{L}$.

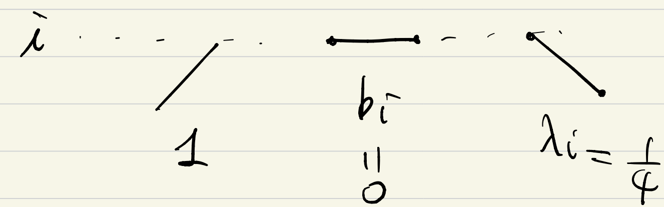
$$\mu_n = \mathcal{L}(x^n) = \frac{2}{\pi}\int_{-1}^{1} x^n (1-x^2)^{\frac{1}{2}}\, dx$$

$$\Rightarrow \mu_{2n} = \frac{1}{4^n} C_n, \quad \mu_{2n+1} = 0.$$

$\hookrightarrow$ Catalan # $= \frac{1}{n+1}\binom{2n}{n}$

Recall

$$\mu_n = \sum_{\pi \in Mot_n} wt(\pi)$$



$$\mu_n = \sum_{\pi \in Dyck_n} \left(\frac{1}{4}\right)^{\frac{n}{2}}$$

$$\Rightarrow \mu_{2n} = \sum_{\pi \in Dyck_{2n}} \frac{1}{4^n}, \quad \mu_{2n+1} = 0$$

$$= C_n / 4^n$$

The Tchebyshev poly of 1st kind

$T_n(x) = \cos n\theta, \qquad x = \cos\theta.$

$$\int_{-1}^{1} T_m(x) T_n(x) (1-x^2)^{-1/2} dx = \pi \delta_{m,n}.$$

$\Rightarrow \{T_n(x)\}_{n\geq 0}$ is OPS for $\mathcal{L}$

$$\mathcal{L}(f(x)) := \frac{1}{\pi}\int_{-1}^{1} f(x)(1-x^2)^{-\frac{1}{2}} dx.$$

$$\mu_n = \mathcal{L}(x^n) = \frac{1}{\pi}\int_{-1}^{1} x^n (1-x^2)^{-\frac{1}{2}} dx$$

Using calculus,

$\mu_{2n} = \frac{1}{4^n}\binom{2n}{n}, \qquad \mu_{2n+1} = 0.$

The monic Tchebyshev 1st kind

$\hat{T}_0(x) = 1, \quad \hat{T}_n(x) = 2^{1-n} T_n(x) \quad (n \geq 1)$

Recall

$$\hat{T}_{n+1}(x) = (x - b_n)\hat{T}_n(x) - \lambda_n \hat{T}_{n-1}(x)$$

$$b_n = 0, \quad \lambda_n = \begin{cases} \frac{1}{2} & \text{if } n = 1 \\ \frac{1}{4} & \text{if } n \geq 2. \end{cases}$$

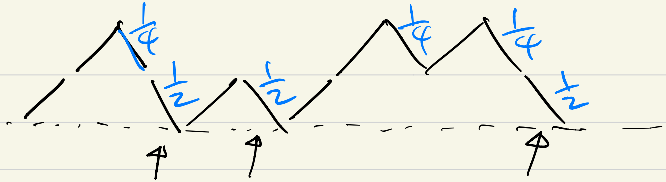
$$\mu_n = \sum_{\pi \in \mathrm{Mot}_n} wt(\pi)$$

$\mu_{2n+1} = 0 \quad (\because$ no horizontal steps$).$

$$\mu_{2n} = \sum_{\pi \in \mathrm{Dyck}_{2n}} \left(\frac{1}{4}\right)^n \cdot 2^{a(\pi)}$$

$a(\pi)$ = # down steps touching x-axis.

e.g.

$\pi =$ 

$a(\pi) = 3$

$wt(\pi) = \left(\frac{1}{4}\right)^n \cdot 2^{a(\pi)}$

We need to show

$$\mu_{2n} = \frac{1}{4^n}\binom{2n}{n} \overset{?}{=} \sum_{\pi \in Dyck_{2n}} \left(\frac{1}{4}\right)^n 2^{a(\pi)}$$

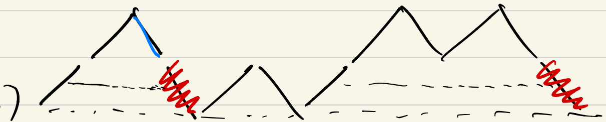
<u>Claim</u> $\displaystyle\sum_{\pi \in Dyck_{2n}} 2^{a(\pi)} = \binom{2n}{n}$.

Pf) $\displaystyle\sum_{\pi \in Dyck_{2n}} 2^{a(\pi)} = \#$ colored Dyck paths from $(0,0)$ to $(2n,0)$.

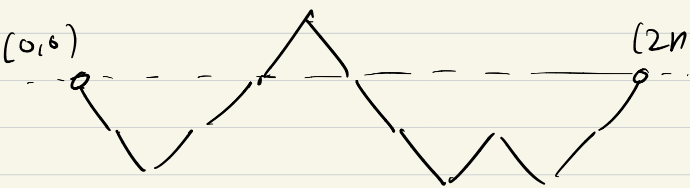
A colored Dyck path means a Dyck path where every down step touching x-axis may or may not be colored red.

n U's

n D's



1-1 $\updownarrow$ flip the part with red color.



$\#$ paths from $(0,0)$ to $(2n,n)$

$= \binom{2n}{n}$. $\square$

§5.2. Hermite polynomails

Def) The Hermite polynomials $H_n(x)$ are defined by $H_{-1}(x)=0$, $H_0(x)=1$,

① $\cdots$ $H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$, $n \geq 0$.

lead coeff of $H_n(x)$ is 2^n.

Define $\hat{H}_n(x) = 2^{-n} H_n(x)$.

Divide ① by 2^{n+1}

$$\hat{H}_{n+1}(x) = x\hat{H}_n(x) - \frac{n}{2}\hat{H}_{n-1}(x).$$

Lem (Rescaling OPS)

Suppose $\{P_n(x)\}_{n\geq 0}$ is a monic OPS with $P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x)$.

Let $\tilde{P}_n(x) = a^n P_n(x/a)$.

Then

$$\tilde{P}_{n+1}(x) = (x - ab_n)\tilde{P}_n(x) - a^2\lambda_n \tilde{P}_{n-1}(x)$$

Pf) Subst. $x \mapsto x/a$ & mult a^{n+1}. $\square$

Def). The rescaled Hermite poly $\tilde{H}_n(x)$ are $\tilde{H}_n(x) = \sqrt{2}^n \hat{H}_n(x/\sqrt{2})$.

$$\tilde{H}_{n+1}(x) = x\tilde{H}_n(x) - n\tilde{H}_{n-1}(x)$$

$$\tilde{H}_{-1} = 0, \quad \tilde{H}_0 = 1.$$

The moments of $\{\tilde{H}_n(x)\}_{n\geq 0}$ are

$$\mu_n = \sum_{\pi \in \mathrm{Motz}_n} \mathrm{wt}(\pi).$$

$b_n = 0, \quad \lambda_n = n.$

$\Rightarrow \quad \mu_{2n+1} = 0.$

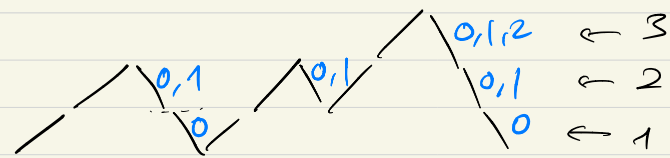
$$\mu_{2n} = \sum_{\pi \in \mathrm{Dyck}_{2n}} \mathrm{wt}(\pi).$$

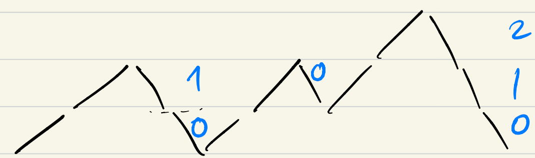
$\left(= (2n-1)!! \right)$
$\#$ complete matchings on $[2n]$.

Q: What does μ_{2n} count?

Def). A Hermite history is a Dyck path where every down step starting at ht k is labeled by an integer in $\{0,1,\ldots,k-1\}$.

$HH_{2n} = \{$Hermite histories from $(0,0)$ to $(2n,0)\}$.





$$\Rightarrow \quad \mu_{2n} = |HH_{2n}|.$$

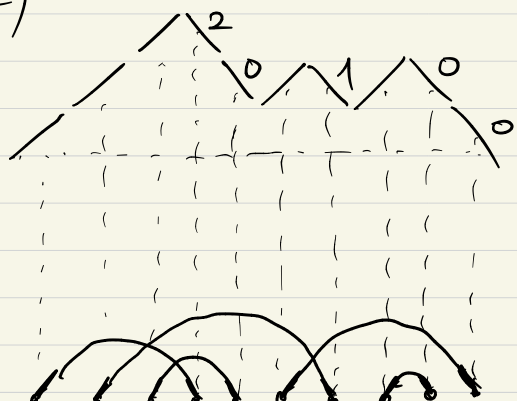
$\because$ $\mathrm{wt}(\pi) = \#$ Hermite histories underlying π.

Let CM_{2n} = set of complete matchings on $[2n]$

We know $|CM_{2n}| = (2n-1)!!$

<u>Prop</u> There is a bijection
$$\phi : HH_{2n} \to CM_{2n}.$$

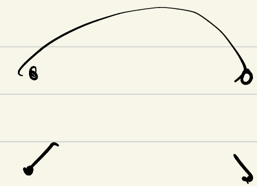
pf)



$\in HH_{10}$

CM_{10}

d
skip 2 openers.



opener closer

ht of starting pt of D

= # available openers. □

<u>Cor</u>

$\underset{\sim}{\mu_{2n}} = |HH_{2n}| = |CM_{2n}| = (2n-1)!!$

(μ_n = # complete matchings on $[n]$.)

classical orthogonal poly

moments μ_n

Hermite # complete matchings on [n]

Charlier # set partitions on [n]

Laguerre # permutations [n].

Jacobi