§5.3. Charlier polynomials.

Def) The (normalied) <u>Charlier poly</u> $C_n(x;a)$ are defined by

$$C_{n+1}(x;a) = (x-n-a)C_n(x;a) - anC_{n-1}(x;a).$$

$$b_n = n+a, \quad \lambda_n = an.$$

Def) A <u>Charlier history</u> a Motzkin path where

$k \cdots$ — $i \in \{0,1,\ldots,k\}$ ($k+1$ choices)

$k \cdots \searrow$ $i \in \{1,\ldots,k\}$ (k choices).

$$CH_n = \{\text{Charlier histories from } (0,0) \text{ to } (n,0)\}$$

If $a=1$,

$$\mu_n = \sum_{\pi \in Motz_n} wt(\pi)$$

$k \cdots$ — $b_k = k+a$

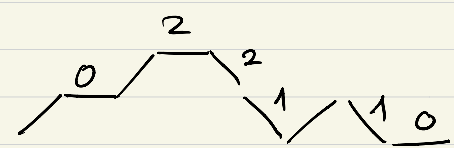
$k \cdots \searrow$ $\lambda_k = ak$

In general,

$$\mu_n = \sum_{\pi \in Motz_n} wt(\pi) = \sum_{p \in CH_n} a^{t(p)}$$

$t(p) = $ (# hori steps with label 0) + (# down steps)

e.g.

$p = $ 

$t(p) = 2+3 = 5$

$\hookrightarrow$ contributes a^5.

$\Pi_n = \{ \text{set partitions of } [n] \}$

For $\sigma \in \Pi_n$, $\mathrm{block}(\sigma) = \#$ blocks in σ.

Thm

$$\mu_n = \sum_{\sigma \in \Pi_n} a^{\mathrm{block}(\sigma)}$$

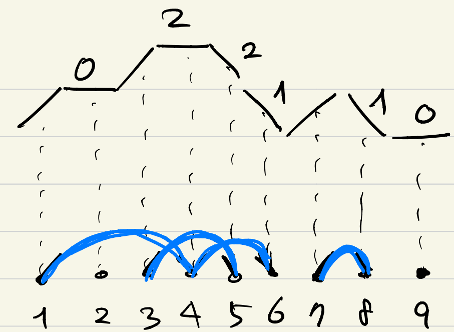
Pf) Sufficient to find a bijection

$\phi : CH_n \longrightarrow \Pi_n$ s.t.

$\rho \longmapsto \sigma \Rightarrow t(\rho) = \mathrm{block}(\sigma)$

$$\therefore \mu_n = \sum_{\rho \in CH_n} a^{t(\rho)} = \sum_{\sigma \in \Pi_n} a^{\mathrm{block}(\sigma)}$$

$\rho =$



$\{ \{1,4,6\}, \{2\}, \{3,5\}, \{7,8\}, \{9\} \} \in \Pi_9$

$/ \longleftrightarrow$ opener

$\backslash \longleftrightarrow$ closer

$\underline{0} \longleftrightarrow \bullet$ singleton

$\underline{i \neq 0} \longleftrightarrow$ transient

$\square$

<u>observation</u> : height of each step is # available open arcs.

§5.4. Laguerre polynomials.

Def) The (normalized) Laguerre poly $L_n^{(\alpha)}(x)$ are defined by

$$L_{n+1}^{(\alpha)}(x) = (x-2n-\alpha)\, L_n^{(\alpha)}(x) - n(n-1+\alpha)\, L_{n-1}^{(\alpha)}(x).$$

$$b_n = 2n+\alpha, \quad \lambda_n = n(n-1+\alpha).$$

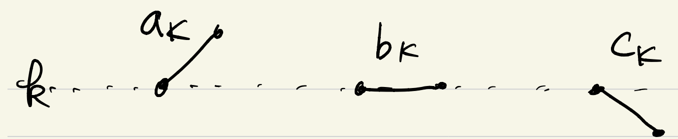
Lem) Suppose $\{P_n(x)\}_{n\geq 0}$ is a monic OPS s.t.

$$P_{n+1}(x) = (x-b_n)P_n(x) - a_{n-1}c_n P_{n-1}(x).$$

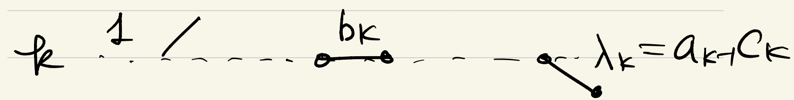
Then

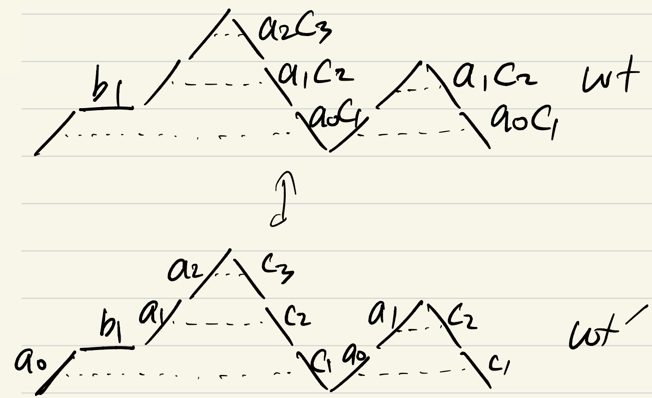
$$\mu_n = \sum_{\pi \in \mathrm{Motz}_n} \mathrm{wt}'(\pi)$$

$\mathrm{wt}'(\pi)$ = product of weights of steps in π.



Pf) $\mu_n = \sum_{\pi \in \mathrm{Motz}_n} \mathrm{wt}(\pi)$





$\square$

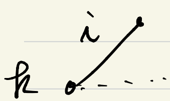
For Laguerre,

$b_n = 2n+\alpha$, $\quad \lambda_n = \underbrace{n}_{= c_n}\underbrace{(n-1+\alpha)}_{= a_{n-1}}$.

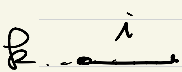
$$\mu_n = \sum_{\pi \in \mathrm{Motz}_n} wt'(\pi)$$

$a_k = k+\alpha$, $\quad b_k = 2k+\alpha$, $\quad c_k = k$.

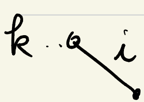
Def) A <u>Laguerre history</u> is a Motzkin path where



$i \in \{0, 1, \dots, k\}$.



$i \in \{-k, \dots, -1, 0, 1, \dots, k\}$



$i \in \{1, \dots, k\}$.

LH_n = set of Laguerre histories from $(0,0)$ to $(n,0)$.

For $\rho \in LH_n$, $\mathrm{zero}(\rho)$ = # steps with label 0.

$$\Rightarrow \quad \mu_n = \sum_{\rho \in LH_n} \alpha^{\mathrm{zero}(\rho)}.$$

Goal: Find a bijection

$\phi : LH_n \to S_n$ s.t.

$\rho \mapsto \sigma$

$\mathrm{zero}(\rho) = \mathrm{cycle}(\sigma)$

This will imply

$$\mu_n = \sum_{\sigma \in S_n} \alpha^{\mathrm{cycle}(\sigma)} = \alpha(\alpha+1)\cdots(\alpha+n-1)$$

$\boxed{\text{Bijection 1}}$ Françon–Viennot.

$\phi : LH_n \to S_n$.

Let $\rho \in LH_n$.

For $k = 0, 1, 2, \dots, n$, we will construct

$A_0, A_1, \dots, A_n$: lists of cycles.

$A_0 = \phi$. Suppose A_{k-1} is constructed.

Case I : kth step is U with label l_k

If $l_k = 0 \Rightarrow A_k = (k\,\bullet)\, A_{k-1}$ new cycle

If $l_k = i > 0 \Rightarrow$

ith dot

$A_{k-1} = (\ \bullet\ )\ \ \bullet\ \ \circ\ \ \bullet$

$\downarrow$

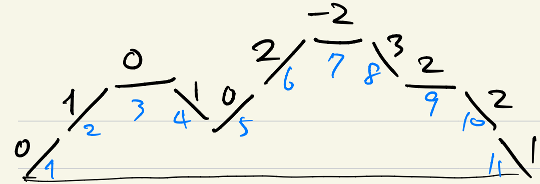
$\bullet\, k\, \bullet$

Case II : kth step $=$ H. with label l_k.

If $l_k = 0$: $A_k = (k)\, A_{k-1}$ new cycle

If $l_k = i > 0$: ith dot $\mapsto k\,\bullet$

If $l_k = -i < 0$: " $\mapsto \bullet\, k$



$A_0 = \phi$ (# 0's = starting ht)

$A_1 = (1\,\bullet)$

$A_2 = (1\,\bullet\,2\,\bullet)$

$A_3 = (3)(1\,\bullet\,2\,\bullet)$

$A_4 = (3)(1\,4\,2\,\bullet)$

$A_5 = (5\,\bullet)(3)(1\,4\,2\,\bullet)$

$A_6 = (5\,\bullet)(3)(1\,4\,2\,\bullet\,6\,\bullet)$

$A_7 = (5\,\bullet)(3)(1\,4\,2\,\bullet\,7\,6\,\bullet)$

$A_8 = (5\,\bullet)(3)(1\,4\,2\,\bullet\,7\,6\,8)$

$A_9 = (5\,\bullet)(3)(1\,4\,2\,9\,\bullet\,7\,6\,8)$

$A_{10} = (5\,\bullet)(3)(1\,4\,2\,9\,10\,7\,6\,8)$

$A_{11} = (5\ 11)(3)(1\ 4\ 2\ 9\ 10\ 7\ 6\ 8)$

Case III kth step $=$ D with label $l_k = i > 0$.

replace ith dot by k

If $\pi \in S_n$

we express π in cycle notation uniquely s.t.

① every cycle starts with min elt.

② cycles are ordered so that min elts are decreasing.

<u>Note</u> The bijection $\phi : LH_n \to S_n$ contains bijection

$$\phi_1 : CH_n \to \Pi_n$$

$$\phi_2 : HH_n \to CM_n.$$

$CH_n \subset LH_n$ In this sense

In $\rho \in LH_n$, every $/$ has wt 0

every $—$ has wt ≥ 0

$\backslash$ same.

In this case every cycle is in increasing order

By making cycles to blocks we get $\sigma \in \Pi_n$.

$HH_n \subset CH_n \subset LH_n$.

where $/$ has wt 0

$—$ X

$\backslash$ same.

$\Rightarrow$ every cycle has length 2.

$\Rightarrow$ complete matching.

$\boxed{\text{Bijection 2}}$ Foata–Zeilberger.

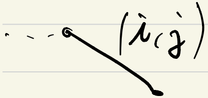
We use the original $wt(\pi)$ for $\pi \in Motz_n$.

Assume $\alpha = 1$.

$$b_k = 2k+1, \quad \lambda_k = k^2$$

$i \in \{-k, \dots, -1, 0, 1, \dots, k\}$

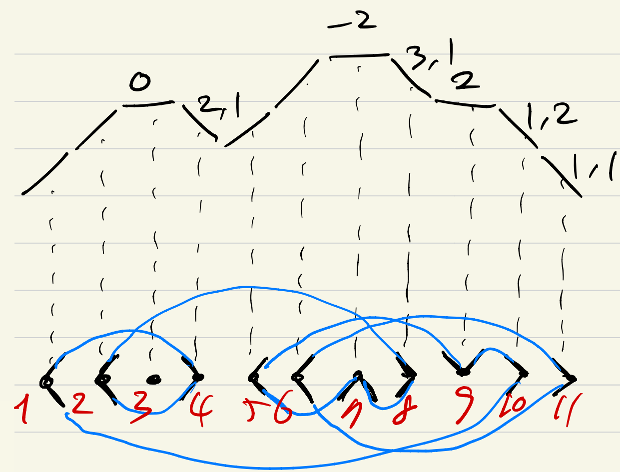
k 

k 

$i, j \in \{1, \dots, k\}$

$\Rightarrow$ Modified Log history.

$M_n = \#\, \text{MLH}.$



$$\pi = \begin{pmatrix} 1 & 2 & \cdots & i & \cdots \\ & & & j & \end{pmatrix}$$