Ch 6. Duality between mixed moments and coefficients.

Suppose $\{P_n(x)\}_{n\geq 0}$ is a monic OPS.

$$P_n(x) = \sum_{k=0}^{n} \nu_{n,k} x^k.$$

We will show

$$x^n = \sum_{k=0}^{n} \mu_{n,k} P_k(x),$$

$$\mu_{n,k} = \frac{\mathcal{L}(x^n P_k(x))}{\mathcal{L}(P_k(x)^2)}$$: mixed moment.

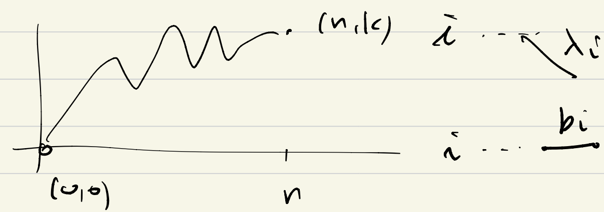
§6.1. Mixed moments and coefficients

$\{P_n(x)\}_{n\geq 0}$: OPS.

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x).$$

$$\mu_{n,k} = \frac{\mathcal{L}(x^n P_k(x))}{\mathcal{L}(P_k(x)^2)} = \sum_{\pi \in \mathrm{Motz}_{n,k}} \mathrm{wt}(\pi)$$

$\mathrm{Motz}_{n,k}$ = set of Motzkin paths from $(0,0)$ to (n,k)



Prop $x^n = \sum_{k=0}^{n} \mu_{n,k} P_k(x)$

Pf) $x^n = \sum_{k=0}^{n} \sigma_{n,k} P_k(x)$.

Multiply $P_k(x)$ & take $\mathcal{L}$.

$$\mathcal{L}(x^n P_k(x)) = \sigma_{n,k} \mathcal{L}(P_k(x)^2)$$

$$\Rightarrow \sigma_{n,k} = \frac{\mathcal{L}(x^n P_k)}{\mathcal{L}(P_k^2)} = \mu_{n,k}.$$

□.

We have

$$x^n = \sum_{k=0}^{n} \mu_{n,k} P_k(x)$$

$$P_n(x) = \sum_{k=0}^{n} \nu_{n,k} x^k.$$

$\{x^n\}_{n\geq 0}$, $\{P_n(x)\}_{n\geq 0}$

are bases of poly space.

$$\begin{pmatrix} x^0 \\ x^1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \mu_{n,k} \end{pmatrix} \begin{pmatrix} P_0 \\ P_1 \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} P_0 \\ P_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \nu_{n,k} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ \vdots \end{pmatrix}$$

$\Rightarrow$ Matrix identities

$$(\mu_{n,k})_{n,k\geq 0} (\nu_{n,k})_{n,k\geq 0} = I$$

$$(\nu_{n,k})_{n,k\geq 0} (\mu_{n,k})_{n,k\geq 0} = I.$$

$\Rightarrow \forall n, m \geq 0$

$$\sum_k \mu_{n,k} \nu_{k,m} = \delta_{n,m}$$

$$\sum_k \nu_{n,k} \mu_{k,m} = \delta_{n,m}$$

§6.2. Combinatorial proof of duality

Let $\{b_n\}_{n\geq 0}$, $\{\lambda_n\}_{n\geq 1}$ be any sequences. (λ_n may be zero).

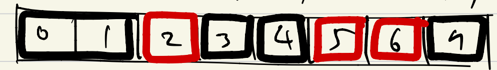
Define

$$\mu_{n,k} = \sum_{\pi \in \mathrm{Motz}_{n,k}} \mathrm{wt}(\pi)$$

$$\nu_{n,k} = \sum_{T \in FT_{n,k}} \mathrm{wt}'(\pi)$$

$FT_{n,k} = \{$ Favard tilings of 1 [ ] n with exactly k red monomines $\}$.

ex). $-\lambda_1$ $\quad -b_3\ -b_4 \quad -b_7$

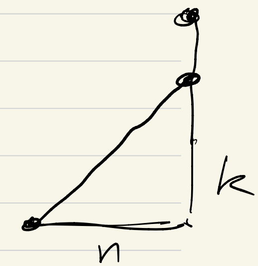
$T =$  $\in FT_{8,3}$

$\mathrm{wt}'(T) = \lambda_1 b_3 b_4 b_7$.

Note If $n<k$,

$$\mu_{n,k} = 0$$

$$\nu_{n,k} = 0$$



Thm $n, m \in \mathbb{Z}_{\geq 0}$.

$$\sum_{k\geq 0} \nu_{n,k}\mu_{k,m} = \delta_{n,m}$$

Pf) We may assume $n \geq m$.

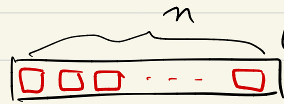
($\because$ $\nu_{n,k}\mu_{k,m} \neq 0$ only if $n \geq k \geq m$)

$(n \geq m)$

$$X = \left\{ (T, \pi) : \begin{array}{l} T \in FT_{n,k} \\ \pi \in Motz_{k,m} \\ m \leq k \leq n \end{array} \right\}$$

$Y = \{ (T, \pi) \in X :$ $FT_{n,n}$

$T =$



$\pi =$

$m \in Motz_{m,m}$ $\}$

If $n > m$, $Y = \phi$.

If $n = m$, Y has a unique pair.

$$\Rightarrow wt'(T) wt(\pi) = 1$$

If we can find a sign-reversing involution $\phi : X \to X$ with fixed point set Y, we are done.

$$\sum_{k \geq 0} \nu_{n,k} \mu_{k,m} = \sum_{(T,\pi) \in X} wt'(T) wt(\pi)$$

$$= \sum_{(T,\pi) \in Y} wt'(T) wt(\pi)$$

$$= \delta_{n,m}.$$

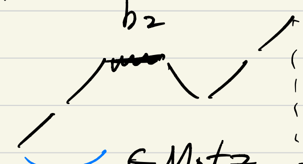
Suppose $(T, \pi) \in X$. $T \in FT_{n,k}$, $\pi \in Motz_{k,m}$.

We define $\phi(T, \pi) = (T', \pi')$.

$$\pi = S_1 S_2 \cdots S_k$$

e.g.

$\pi =$

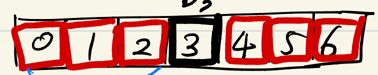


$\in Motz_{6,3}$

$u=2$

$- b_3$

$T =$



$a=3$

$\in FT_{7,6}$

$u =$ max # upsteps at beginning

$a =$ max # red mono at beginning.

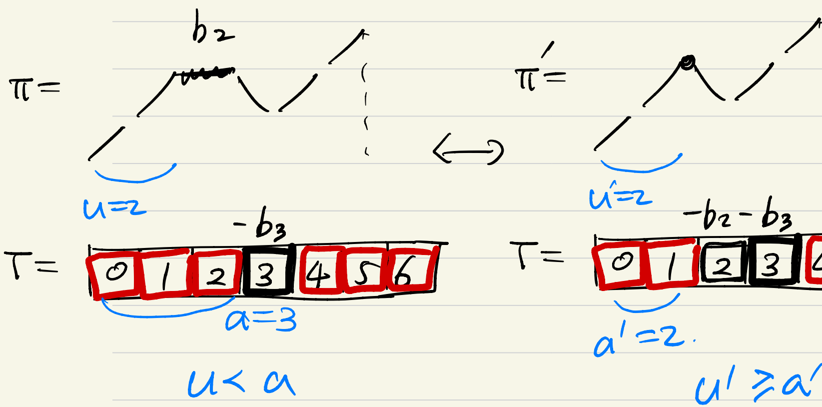
Compare a & u.

case I : $u < a$.

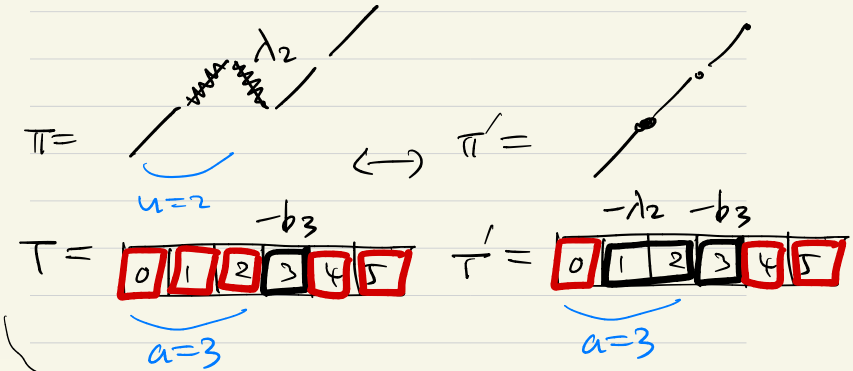
case I-1 $S_{u+1} = H$.

$\pi' = \pi \setminus S_{u+1}$

T' = make the $(u+1)$st red monomino into black ".



case I-2. $S_{u+1} = D$.

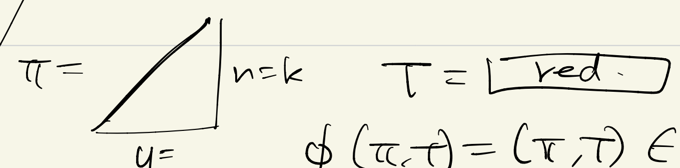


Case II : $u \geq a \neq n$.

Similar to case I except we go reverse direction.

case III : $u \geq a = n$.

$u \leq k \leq n$. $\Rightarrow$ $u = a = n$.

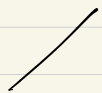
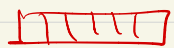


$\phi(\pi, T) = (\pi, T) \in Y$. $\square$

Thm $\sum_{k\geq 0} \mu_{n,k} \nu_{k,m} = \delta_{n,m}$.

Pf) We may assume $n \geq m$.

$$X = \left\{ (\pi, T) : \begin{array}{l} \pi \in \mathrm{Motz}_{n,k} \\ T \in \mathrm{FT}_{k,m} \\ m \leq k \leq n \end{array} \right\}$$

$Y = \{ (\pi, T) \in X :$ $\pi =$  , $T =$  $\}$

$$\sum_{k\geq 0} \mu_{n,k} \nu_{k,m} = \sum_{(\pi,T)\in X} \mathrm{wt}(\pi)\, \mathrm{wt}'(T)$$

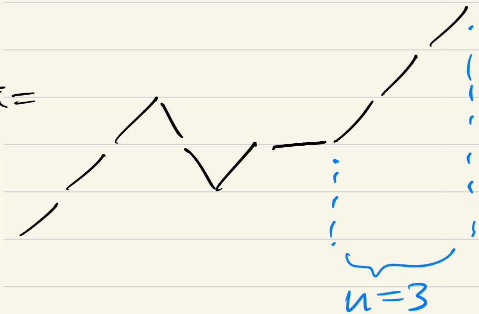
Claim: $\exists$ sign-reversing weight-pres. involution $\phi : X \to X$ with $\mathrm{Fix}\,\phi = Y$.

Suppose $(\pi, T) \in X$. $\pi \in \mathrm{Motz}_{n,k}$, $T \in \mathrm{FT}_{n,m}$.

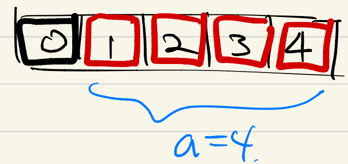
let $\pi = S_n \cdots S_1$

e.g.

$\pi =$



$T =$

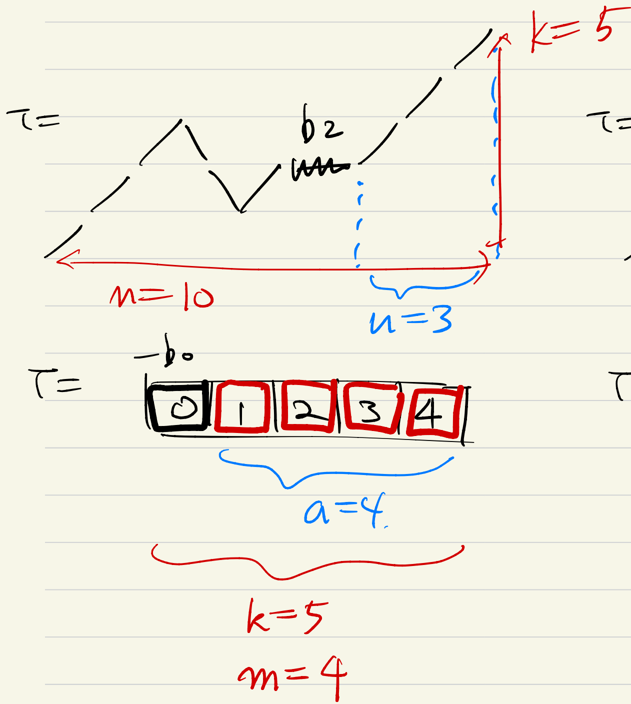


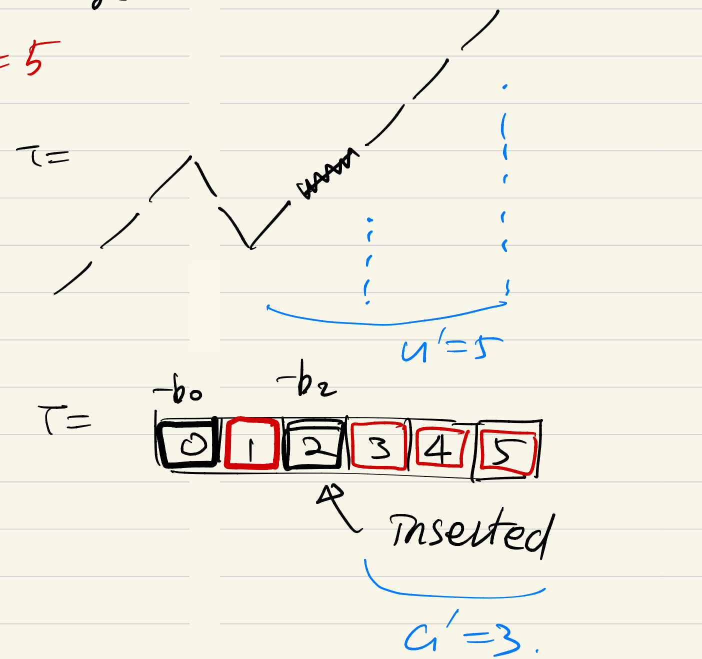
Define $\phi(\pi, T) = (\pi', T')$.

case I-2 $S_{u+1} = D$.

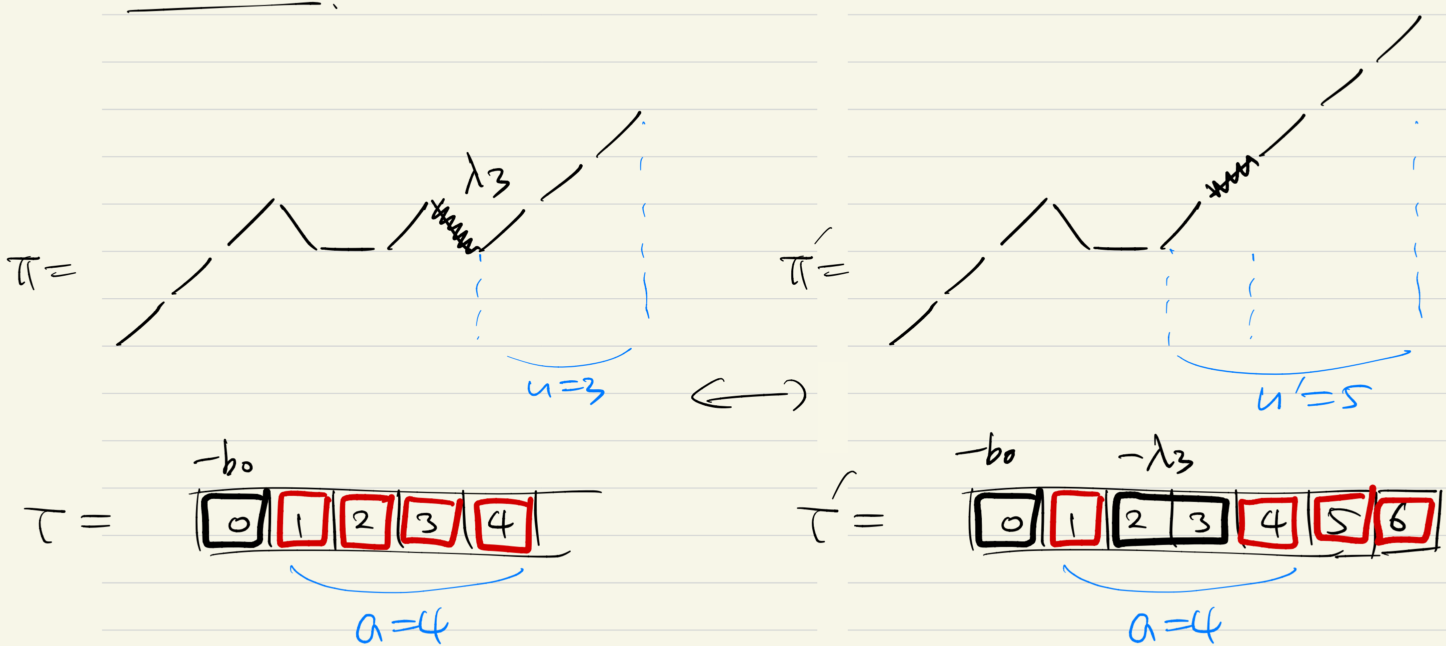
case I $n \neq u \leq a$.

case I-1 $S_{u+1} = H \rightarrow$ Change this to U.





case I-2 $S_{u+1} = D$.



Case 2 $u > a$.

case 3 $n = u \leq a$. $\phi(\pi, \tau) = (\pi, \tau) \in Y$. ∎

$\nRightarrow$ $n = u = a$.