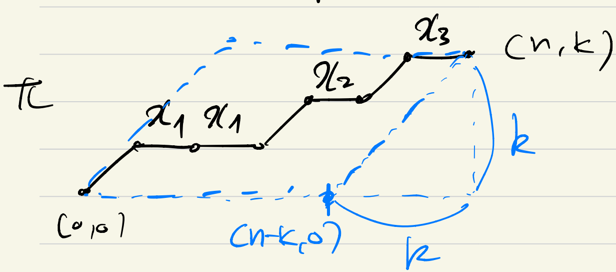
§6.3. Special case: elementary and homogeneous symmetric functions.

let $b_k = x_k$ (indeterminates)

$\lambda_k = 0$.

Since $\lambda_k = 0$

if $\pi \in Motz_{n,k}$ then only U & H steps.

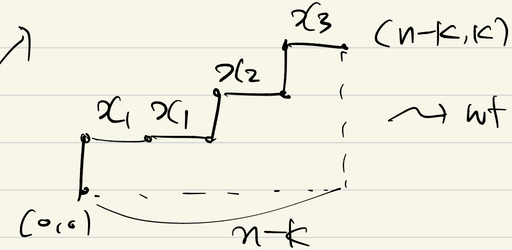


If we define $U' = (0,1)$

then

π can be identified

with $\pi' : (0,0) \to (n-k, k)$



$\rightsquigarrow wt = x_1 x_1 x_2 x_3$

$$\mu_{n,k} = \sum_{\pi : (0,0) \to (n-k,k)} \underbrace{wt(\pi)}$$

prod of x_i

for i

<u>Note</u>: π is determined by $wt(\pi)$

π is determined by ht of hor steps.

every wt looks like

$$x_{i_1} x_{i_2} \cdots x_{i_{n-k}}$$

$$0 \leq i_1 \leq i_2 \leq \cdots \leq i_{n-k} \leq k$$

$$\mu_{n,k} = \sum_{0 \leq i_1 \leq \cdots \leq i_{n-k} \leq k} x_{i_1} \cdots x_{i_{n-k}}.$$

Def). Let $x = (x_0, x_1, x_2, \dots)$.
A power series $f(x_0, x_1, \dots)$ in the variables $x_0, x_1, \dots$ is called a <u>symmetric function</u> if it is invariant under permuting variables.

e.g. $f(x_0, x_1, \dots) = x_0 + x_1 + \cdots$ sym ftn.

$f(x_0, x_1, \dots) = x_0 + x_1^2$
$\neq x_1 + x_0^2$
↳ not sym.

A <u>homogeneous symmetric function</u> (complete) in $x_0, x_1, \dots$ is

$$h_k = \sum_{i_1 \le \cdots \le i_k} x_{i_1} \cdots x_{i_k}.$$

An <u>elementary symmetric function</u> is

$$e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}.$$

A <u>homogeneous sym poly</u> is

$$h_k(x_0, \dots, x_n) = \sum_{0 \le i_1 \le \cdots \le i_k \le n} x_{i_1} \cdots x_{i_k}$$

An <u>elementary sym poly</u> is

$$e_k(x_0, \dots, x_n) = \sum_{0 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$$

Define $h_0 = e_0 = 1$
and $h_k = e_k = 0$ if $k < 0$.

ex). $h_1(x_0, x_1, x_2) = e_1(x_0, x_1, x_2) = x_0 + x_1 + x_2$
$h_2(x_0, x_1, x_2) = x_0^2 + x_1^2 + x_2^2 + x_0 x_1 + x_0 x_2 + x_1 x_2$
$e_2(\quad " \quad) = x_0 x_1 + x_0 x_2 + x_1 x_2.$
$e_3(x_0, x_1, x_2) = x_0 x_1 x_2$
$e_4(x_0, x_1, x_2) = 0.$

Thm. If $b_k = x_k$, $\lambda_k = 0$, then

$$\mu_{n,k} = h_{n-k}(x_0, x_1, \dots, x_k)$$

$$\nu_{n,k} = \sum_{T \in FT_{n,k}} wt'(T).$$

Since $\lambda_k = 0$, no dominos!

$\Rightarrow$ T has $\square$, $\square$ only.

ex)

$$T = \begin{array}{ccccccc} -x_0 & & -x_2 & -x_3 & & -x_5 & \\ \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} & \boxed{4} & \boxed{5} & \boxed{6} \end{array}$$

wt is always $(-1)^{n-k} x_{i_1} \cdots x_{i_{n-k}}$

$0 \le i_1 < \cdots < i_{n-k} \le n-1$.

$$\Rightarrow \nu_{n,k} = (-1)^{n-k} \sum_{0 \le i_1 < \cdots < i_{n-k} \le n-1} x_{i_1} \cdots x_{i_{n-k}}$$

Thm $\nu_{n,k} = (-1)^{n-k} e_{n-k}(x_0, \dots, x_{n-1})$.

Cor

$$\left(h_{n-k}(x_0, \dots, x_k)\right)_{n,k \ge 0} \left((-1)^{n-k} e_{n-k}(x_0, \dots, x_{n-1})\right)_{n,k \ge 0} = I$$

$$\left((-1)^{n-k} e_{n-k}(x_0, \dots, x_{n-1})\right)_{n,k \ge 0} \left(h_{n-k}(x_0, \dots, x_k)\right)_{n,k \ge 0} = I$$

One equivalent identity is

$n, m \geq 0$ fixed.

$$\sum_{k\geq 0} (-1)^{n-k} e_{n-k}(x_0, \dots, x_{n-1}) \cdot h_{k-m}(x_0, \dots, x_m) = \delta_{n,m}.$$

let $N = n-m \geq 0$.

$$\sum_{k\geq 0} (-1)^{N+m-k} e_{N+m-k}(x_0, \dots, x_{N+m-1}) \cdot h_{k-m}(x_0 \dots x_m) = \delta_{N+m,m}$$

$k \mapsto m+k$.

$$\sum_{k\geq 0} (-1)^{k} e_{N-k}(x_0, \dots, x_{N+m-1}) h_k(x_0 \dots x_m) = \delta_{N,0}$$

let $m \to \infty$.

$$\Rightarrow \sum_{k=0}^{N} (-1)^k e_{N-k} h_k = \delta_{N,0}.$$

well known.

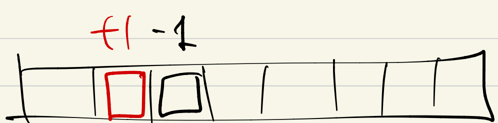
§6.4. Special case: binomial coefficients

Let $b_k = 1$, $\lambda_k = 0$.

$\mu_{n,k} =$

$= \binom{n}{k}$.

$$\nu_{n,k} = \sum_{T \in FT_{n,k}} wt'(T)$$



$wt' = (-1)^{n-k}$

$\nu_{n,k} = (-1)^{n-k} \binom{n}{k}$.

<u>Thm</u> If $b_k = 1$, $\lambda_k = 0$

$\Rightarrow \mu_{n,k} = \binom{n}{k}$, $\nu_{n,k} = (-1)^{n-k} \binom{n}{k}$.

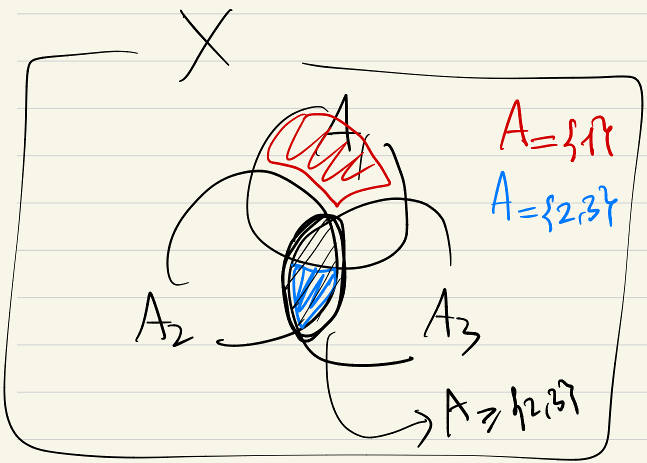
<u>Cor</u> $\left[\left(\binom{n}{k} \right)_{n,k=0}^{\infty} \right]^{-1} = \left((-1)^{n-k} \binom{n}{k} \right)_0^{\infty}$

$$\begin{pmatrix} \binom{0}{0} & & & \\ \binom{1}{0} & \binom{1}{1} & & 0 \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \\ & \vdots & & \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \\ & \vdots & & \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ 1 & -2 & 1 & \\ -1 & 3 & -3 & 1 \\ & \vdots & & \end{pmatrix}$$

Let X be a set (finite).

Let $A_1, \dots, A_n \subseteq X$.



For $I \subseteq [n]$, define

$$A_{=I} = \{x \in X \mid x \in A_i \iff i \in I\}$$

$$A_{\geq I} = \{x \in X \mid x \in A_i \text{ for all } i \in I\}$$

$$= A_{i_1} \cap \cdots \cap A_{i_k} \quad (I = \{i_1, \dots, i_k\})$$

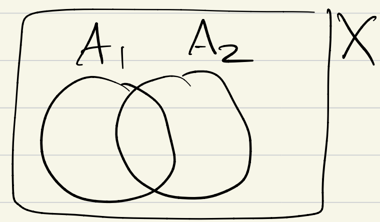
$$A_{\geq I} = \bigcap_{i \in I} A_i = \bigcup_{J \supseteq I} A_{=J}.$$

$$\Rightarrow |A_{\geq I}| = \sum_{J \supseteq I} |A_{=J}|$$

Thm (Principle of inc-ex.)

$$|A_{=I}| = \sum_{J \supseteq I} (-1)^{|J-I|} |A_{\geq J}|$$

e.g.



$$|A_{=\emptyset}| = |A_1^c \cap A_2^c| = + |A_{\geq \emptyset}| - |A_{\geq\{1\}}| - |A_{\geq\{2\}}| + |A_{\geq\{1,2\}}|$$

$$= |X| - |A_1| - |A_2| + |A_1 \cap A_2|.$$

Thm (Principle of inc - ex.)

$$|A_{=I}| = \sum_{J \supseteq I} (-1)^{|J-I|} |A_{\geq J}|$$

pf). We consider every $x \in X$ and compute its contribution to LHS & RHS.

Let $x \in X$.

Suppose $x \in A_{=K}$.

Case I $K = I$.

total contrib of x to LHS $= +1$

total " RHS

$$\hookrightarrow = \sum_{\substack{J \supseteq I \\ x \in A_{\geq J}}} (-1)^{|J-I|} = \sum_{I \subseteq J \subseteq K} (-1)^{|J-I|} = 1$$

Case II $K \supsetneq I$

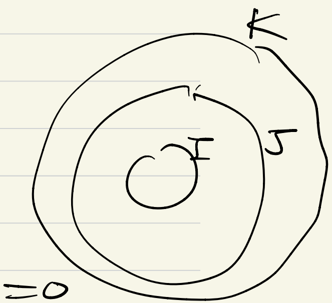
total contrib of x to LHS $= 0$

" RHS

$$= \sum_{I \subseteq J \subseteq K} (-1)^{|J-I|}$$

$$= \sum_{J \subseteq K-I} (-1)^{|J|}$$

$$= \sum_{j=0}^{|K-I|} \binom{|K-I|}{j} (-1)^{j} = 0$$



Case III $K \not\supseteq I$.

cont to LHS $= 0$

" $= 0$.

$\square$.

If $I = \phi$,

$|A_{=\phi}| = |A_1^c \cap \cdots \cap A_n^c|$

$= |X| - |A_1| - \cdots - |A_n|$

$+ |A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|$

$\vdots$

$+ (-1)^n |A_1 \cap \cdots \cap A_n|$.

(usual principle of IE).

ex) A <u>derangement</u> is a permutation $\pi \in S_n$ s.t. $\pi(k) \neq k \quad \forall k$.

(no fixed pts).

$X = S_n$.

$A_i = \{\pi \in S_n \mid \pi(i) = i\}$.

$d_n = \#\text{ derangements} = |A_{=\phi}|$

$$= \sum_{J \subseteq [n]} (-1)^{|J|} |A_{\geq J}|$$

If $|J| = j$, $|A_{\geq J}| = (n-j)!$

$$d_n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)!$$

$$= n! \sum_{j=0}^{n} (-1)^j \frac{1}{j!}$$

$$\frac{d_n}{|S_n|} = \sum_{j=0}^{n} (-1)^j \frac{1}{j!} \approx \frac{1}{e}$$

Suppose $|A_{\geq I}|$ & $|A_{=I}|$ depend only on $|I|$.

We can write, if $|I|=k$,

$$|A_{\geq I}| = b_k, \quad |A_{=I}| = a_k.$$

$$|A_{\geq I}| = \sum_{J \supseteq I} |A_{=J}|$$

$$|A_{=I}| = \sum_{J \supseteq I} (-1)^{|J-I|} |A_{\geq J}|$$

$$b_k = \sum_{j=k}^{n} \binom{n-k}{j-k} a_j$$

$$a_k = \sum_{j=k}^{n} (-1)^{j-k} \binom{n-k}{j-k} b_j$$

Let $a'_k = a_{n-k}$, $b'_k = b_{n-k}$.

We can rewrite these as

$$b'_k = \sum_{j=0}^{k} \binom{k}{j} a'_j$$

$$a'_k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} b'_j$$

$$\begin{pmatrix} b'_0 \\ \vdots \\ b'_n \end{pmatrix} = \left(\binom{i}{j} \right)_{i,j=0}^{n} \begin{pmatrix} a'_0 \\ \vdots \\ a'_n \end{pmatrix}$$

$$\begin{pmatrix} a'_0 \\ \vdots \\ a'_n \end{pmatrix} = \left((-1)^{i-j} \binom{i}{j} \right)_{i,j=0}^{n} \begin{pmatrix} b'_0 \\ \vdots \\ b'_n \end{pmatrix}$$

$$\Rightarrow \left(\binom{i}{j} \right)^{-1} = \left((-1)^{i-j} \binom{i}{j} \right)$$