§ 6.5. Special case: q-binomial coefficients.

Def) For int $n \geq 0$, the <u>q-integer</u> $[n]_q$ is

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1}.$$

If $q=1$, then $[n]_q = n$.

The <u>q-factorial</u> $[n]_q!$ is

$$[n]_q! = [1]_q[2]_q \cdots [n]_q.$$

The <u>q-binomial coefficient</u> $(0 \leq k \leq n)$.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

If $q=1$,

$$[n]_q! = n!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

If $k>n$, we define $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$.

<u>Pascal's identity</u>

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

<u>Lem</u> (Pascal id for q-binom)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$$

Pf) Just computation!

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

$$RHS = q^{n-k}\frac{[n-1]!}{[k-1]!\,[n-k]!} + \frac{[n-1]!}{[k]!\,[n-1-k]!}$$

$$= \frac{q^{n-k}[n-1]!\,[k] + [n-1]!\,[n-k]}{[k]!\,[n-k]!}$$

$$= \frac{[n-1]!}{[k]!\,[n-k]!}\underbrace{\left(q^{n-k}[k]+[n-k]\right)}_{=[n]}.$$

$\square$

Def) A <u>partition</u> is a sequence

$\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of integers

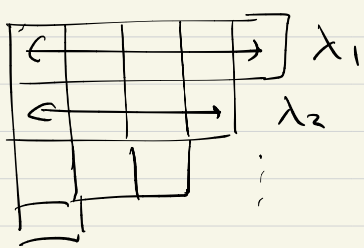
s.t. $\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0$.

Each λ_i is called a <u>part</u>

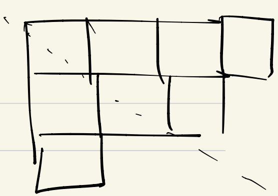
The <u>size</u> of λ is

$|\lambda| = \lambda_1 + \cdots + \lambda_\ell$.

The <u>Young diagram</u> of λ is



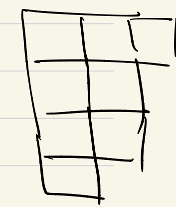
e.g. $\lambda = (4,3,1)$.



The <u>transpose</u> of λ is

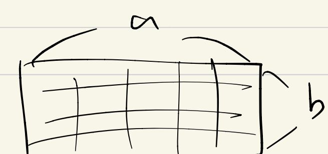
$\lambda' =$ partition whose Young diagram is YD of λ reflected about the main diag.

e.g. If $\lambda = (4,3,1)$, $\lambda' = (3,2,2,1)$.



$$(a^b) := (\underbrace{a, a, \ldots, a}_{b})$$

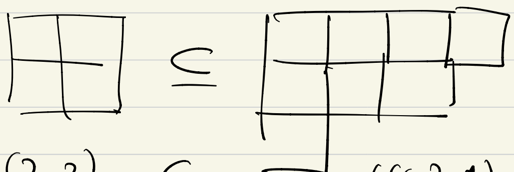
$=$



We write $\mu \subseteq \lambda$

if YD of μ is contained in

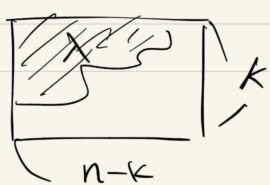
〃 λ.



$(2,2) \subseteq (4,3,1)$.

Prop For $0 \le k \le n$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq ((n-k)^k)} q^{|\lambda|}$$



Pf) Induction on n.

If $n=0$, then $k=0$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1 = \sum_{\lambda \subseteq \emptyset} q^{|\lambda|} = q^{|\emptyset|} = 1.$$

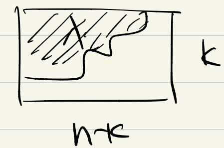
Let $n \geq 1$. Suppose statement holds for $n-1$.

Let $R(n,k)$ be RHS.

Enough to show

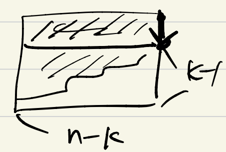
$$R(n,k) = q^{n-k} R(n-1,k-1) + R(n-1,k)$$

$R(n,k)$ = g.f for



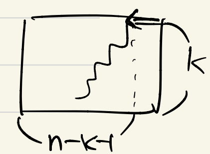
case I

$\lambda_1 = n-k$



Case II

$\lambda_1 < n-k$



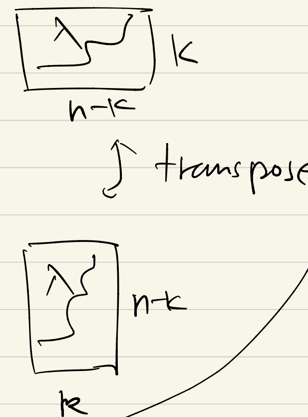
$$q^{n-k} R(n-1,k-1) \quad + \quad R(n-1,k)$$

□

Cor

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq ((n-k)^k)} q^{|\lambda|}$$

$$= \sum_{\lambda \subseteq (k^{n-k})} q^{|\lambda|}$$



$$\mu_{n,k} = \sum_{0 \le i_1 \le \cdots \le i_{n-k} \le k} q^{i_1 + \cdots + i_{n-k}}$$

$$\lambda = (i_{n-k}, \ldots, i_1) \subseteq (k^{n-k})$$

$$= \sum_{\lambda \subseteq (k^{n-k})} q^{|\lambda|} = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Prop If $b_k = q^k$, $\lambda_k = 0$.

then

$$\mu_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q , \quad \nu_{n,k} = (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}.$$

Pf). We have $\mu_{n,k} = h_{n-k}(x_0, x_1, \ldots x_k)$

$$\nu_{n,k} = (-1)^{n-k} e_{n-k}(x_0, \ldots, x_{n-1})$$

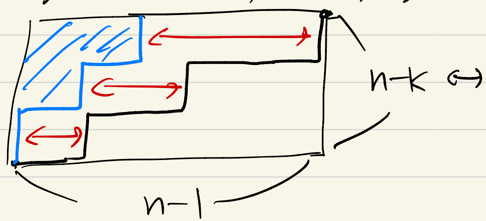
$$x_i = q^i$$

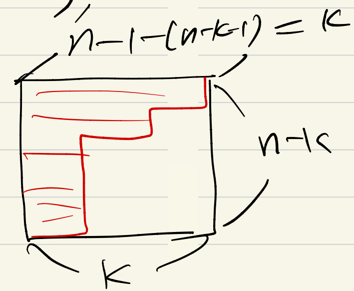
$$(-1)^{n-k} v_{n,k}$$

$$= \sum_{0 \le i_1 < \cdots < i_{n-k} \le n-1} q^{i_1 + \cdots + i_{n-k}}$$

If $\lambda = (i_{n-k}, \ldots, i_1) \subseteq ((n-1)^{n-k})$,

let $\mu = \lambda - (n-k-1, \ldots, 1, 0)$





$$|\lambda| = |\mu| + (0 + 1 + \cdots + (n-k-1))$$

$$= |\mu| = \binom{n-k}{2}$$

$$\sum_{\substack{\lambda \subset ((n-1)^{n-k}) \\ \lambda:\ \text{strictly dec}}} q^{|\lambda|} = \sum_{\mu \subseteq (k^{n-k})} q^{\binom{n-k}{2}} \cdot q^{|\mu|}$$

$$= q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q. \qquad \square$$

§6.6. Special case: Stirling #.

Thm $b_r = r,\ \lambda_r = 0.$

$\Rightarrow \mu_{n,k} = S(n,k)$

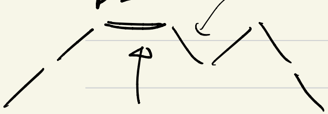
$\nu_{n,k} = s(n,k)$

Pf). Recall if $b_r = r+1,\ \lambda_r = r$

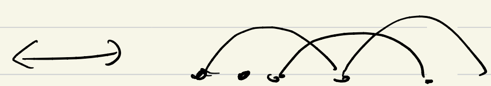
we have Charlier histories

$\mu_n = \#\, CH_n.$

$b_2 = 2+1$ $\quad \lambda_2 = 2$ label $1, 2$



$0, 1, 2$



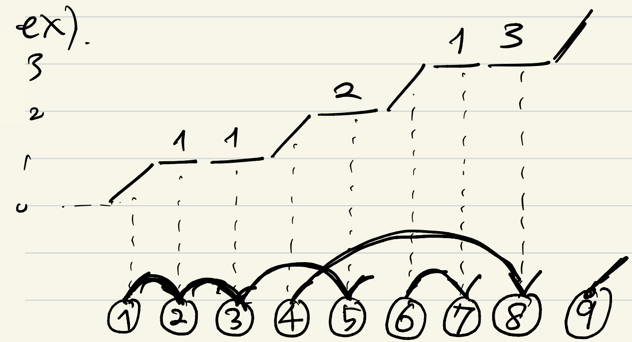
set partitions.

So, if $b_r = r,\ \lambda_r = 0$, then

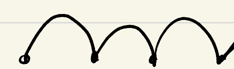
$\mu_{n,k} = \#$ Charlier histories from $(0,0)$ to (n,k) with only / —

every hori i label in $\{1, \dots, i\}$.

ex).



every block is



$S(n,k)$

$\#$ blocks $=$ ending ht $= k$

$\|$

$\#$ such Charlier histories

$= \#$ set partitions of $[n]$ into k bks

Since $b_r = r$, $\lambda_r = 0$,

$$P_n(x) = \underbrace{\sum_{k=0}^{n} \nu_{n,k} x^k}$$

$$= (x-b_0)(x-b_1)\cdots(x-b_{n-1})$$

$$= x(x-1)\cdots(x-n+1)$$

$$= \underbrace{\sum_{k=0}^{n} s(n,k)\, x^k} \leftarrow \left(\because \sum_{\pi \in S_n} \underbrace{x^{\mathrm{cycle}(\pi)}} = x(x+1)\cdots(x+n-1) \right)$$

$$\sum_{\pi \in S_n} x^{\mathrm{cycle}(\pi)} = \sum_k c(n,k)\, x^k$$

$$\Rightarrow \nu_{n,k} = s(n,k)$$

$$P_{n+1} = (x-b_n)P_n - \cancel{\lambda_n P_{n-1}}$$

Ch 7. Determinants of moments.

§7.1. Computing b_n, λ_n using μ_n.

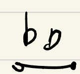
Recall

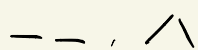
$$\mu_n = \sum_{\pi \in Motz_n} wt(\pi)$$

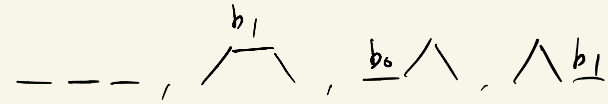
Q: Can we find b_n, λ_n using μ_n only?

A: Yes!

For $n = 1, 2, 3$,

$\mu_1 =$  $= b_0$

$\mu_2 =$  $= b_0^2 + \lambda_1$

$\mu_3 =$ 

$$= b_0^3 + 2b_0\lambda_1 + b_1\lambda_1$$

$b_0 = \mu_1$, $\lambda_1 = \mu_2 - b_0^2 = \mu_2 - \mu_1^2$

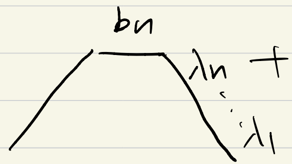
$$b_1 = \frac{\mu_3 - b_0^3 - 2b_0\lambda_1}{\lambda_1}$$

$$= \frac{\mu_3 - \mu_1^3 - 2\mu_1(\mu_2 - \mu_1^2)}{\mu_2 - \mu_1^2}$$

$\vdots$

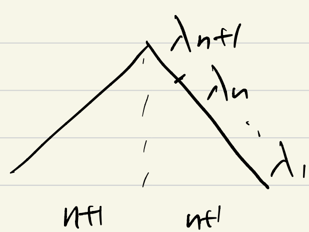
This is always possible!

If $b_0, \ldots, b_{n-1}, \lambda_1, \ldots, \lambda_n$ are known,

$\Rightarrow \mu_{2n+1} =$  + lower terms

$\Rightarrow b_n =$ in terms of μ_i's.

If $b_0, \ldots, b_n, \lambda_1 \ldots \lambda_n$ known

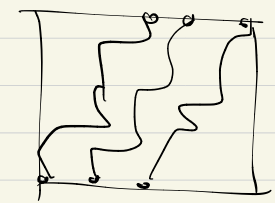
$\Rightarrow \mu_{2n+2} =$  + lower terms.

$\Rightarrow \lambda_{n+1} =$ in terms of μ_i's.

Q: explicit formula?

A: Yes.

We need to build some theory on nonintersecting lattice paths



Lindström–Gessel–Viennot lemma

"Proofs from The Book"