§7.2. The Lindström-Gessel-Viennot Lemma

Def) A **graph** is a pair $G=(V,E)$ of sets V and E,

$E \subseteq V \times V$

Every $v \in V$ is called a vertex

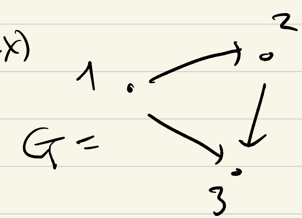
" $e \in E$ " an edge.

If we say G is directed (undirected), it means

$(u,v) \neq (v,u)$ as edges

$((u,v) = (v,u))$

ex)



$G=(V,E)$

$V=\{1,2,3\}$

$E=\{(1,2),(2,3),(1,3)\}$

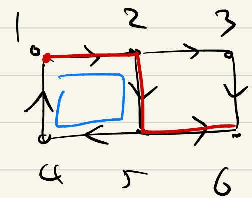
directed.

A path from u to v is a seq of vertices $(v_0, v_1, \ldots, v_n)$ s.t.

$v_0 = u$, $v_n = v$

$(v_i, v_{i+1}) \in E$, $0 \leq i \leq n-1$.

ex)



$p=(1,2,5,6)$ is path from 1 to 6.

A cycle is a path from u to u.

$(1,2,5,4,1)$ is a cycle.

If G has no cycles, G is acyclic.

$P(u \to v)$ = set of paths from u to v.

An **edge weight** of $G=(V,E)$ is a function $\omega : E \to \underbrace{K}_{\text{commutative ring}}$.

The **weight** of a path $p=(v_0, \ldots v_n)$ is $\omega(p) = \omega(v_0, v_1) \cdots \omega(v_{n-1}, v_n)$.

An **n-path** is a sequence of n paths $\mathbb{P} = (p_1, \ldots, p_n)$.

Two paths p_1 and p_2 are **intersecting** if they have a common vertex. Otherwise, **nonintersecting**.

We say $\mathbb{P} = (p_1, \ldots, p_n)$ is nonintersecting if p_i and p_j are nonintersecting for all $i \neq j$.

G: a directed graph with edge weight ω.

Let $\mathbb{A} = (A_1, \ldots, A_n)$, $\mathbb{B} = (B_1, \ldots, B_n)$ be seq of vertices.

$P(\mathbb{A} \to \mathbb{B})$ = set of all n-paths $\mathbb{P} = (p_1, \ldots, p_n)$ s.t. $p_i \in P(A_i \to B_{\sigma(i)})$, $1 \le i \le n$ for some $\sigma \in S_n$.

$\operatorname{sgn}(\mathbb{P}) = \operatorname{sgn}(\sigma)$

$\omega(\mathbb{P}) = \omega(p_1) \cdots \omega(p_n)$.

$NI(\mathbb{A} \to \mathbb{B})$ = set of all nonintersecting n-paths in $P(\mathbb{A} \to \mathbb{B})$.

Thm (Lindström–Gessel–Viennot Lemma. LGV lem)

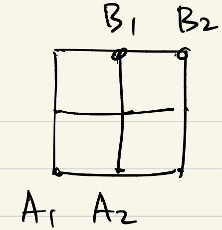
G: a directed acyclic graph with edge weight w.

$\mathbb{A}=(A_1,\dots,A_n)$, $\mathbb{B}=(B_1,\dots,B_n)$.

$$M=(M_{ij})_{i,j=1}^{n}$$

$$M_{ij}=\sum_{p\in P(A_i\to B_j)} w(p)$$

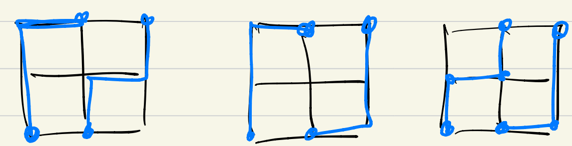
$$\Rightarrow \det M = \sum_{\mathbb{P}\in NI(\mathbb{A}\to\mathbb{B})} \mathrm{sgn}(\mathbb{P})\, w(\mathbb{P}).$$

ex) $G =$ 

$w(e)=1$.

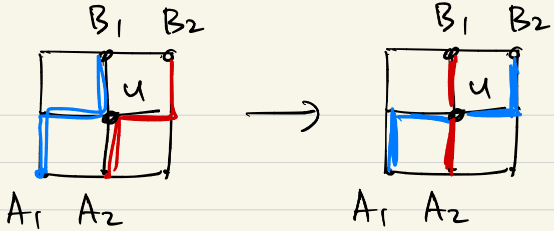
$$M=(M_{ij})_{i,j=1}^{n} = \begin{pmatrix} \binom{3}{1} & \binom{4}{2} \\ \binom{2}{0} & \binom{3}{1} \end{pmatrix}$$

$$\det M = \det\begin{pmatrix} 3 & 6 \\ 1 & 3 \end{pmatrix} = 9-6 = 3.$$



Any $\mathbb{P}\in P(\mathbb{A}\to\mathbb{B})$, $P_1\in P(A_1\to B_1)$, $P_2\in P(A_2\to B_2)$

$$NI(\mathbb{A}\to\mathbb{B}) \subseteq \underbrace{P(A_1\to B_1)\times P(A_2\to B_2)}_{\text{card} = \binom{3}{1}\binom{3}{1}}$$



Let's count # intersecting 2-paths.

$$p_1 \in P(A_1 \to B_1)$$
$$p_2 \in P(A_2 \to B_2)$$

Find first intersection u of p_1, p_2

Let $p_1 = \underbrace{p_1'}_{\text{before } u}\,\underbrace{p_1''}_{\text{after } u}$, $p_2 = p_2' p_2''$

$\Rightarrow$ $\underbrace{q_1 = p_1' p_2'', \quad q_2 = p_2' p_1''}$

p_1, p_2 with tails exchanged.

$\Rightarrow$ $q_1 \in P(A_1 \to B_2)$

$q_2 \in P(A_2 \to B_1)$.

Any such (q_1, q_2) intersect.

We can do the tail-exchange to set (p_1, p_2) back.

This gives a bijection from intersecting (p_1, p_2)

and $\underbrace{P(A_1 \to B_2) \times P(A_2 \to B_1)}$

$$\text{card} = \binom{4}{2} \cdot \binom{2}{0}$$

$$\# NI = \binom{3}{1}\binom{3}{1} - \binom{4}{2}\binom{2}{0}$$

$$= \det \begin{pmatrix} \binom{3}{1} & \binom{4}{2} \\ \binom{2}{0} & \binom{3}{1} \end{pmatrix}$$

Proof of LGV-Lem

$$\det M = \det(M_{ij}) = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)}$$

$$= \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} \sum_{p\in P(A_i\to B_{\sigma(i)})} \omega(p)$$

$$= \sum_{\mathbb{p}\in P(\mathbb{A}\to\mathbb{B})} \mathrm{sgn}(\mathbb{p})\,\omega(\mathbb{p}).$$

?

$$= \sum_{\mathbb{p}\in NI(\mathbb{A}\to\mathbb{B})} \mathrm{sgn}(\mathbb{p})\,\omega(\mathbb{p}).$$

It is enough to find a sign-reversing & weight-pres. involution ϕ on $P(\mathbb{A}\to\mathbb{B})$ with fix pt set $NI(\mathbb{A}\to\mathbb{B})$.

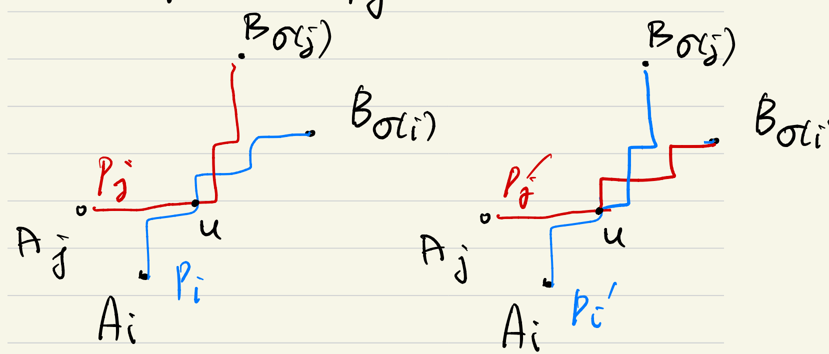
Let $\mathbb{p}=(p_1,\dots,p_n)\in P(\mathbb{A}\to\mathbb{B})$, where $p_i\in P(A_i\to B_{\sigma(i)})$, $\sigma\in S_n$.

If $\mathbb{p}$ is nonintersecting, $\phi(\mathbb{p})=\mathbb{p}$.

Suppose $\mathbb{p}$ is intersecting.

Find the lexicographically smallest (i,j) s.t. p_i and p_j intersect.

Let u be the first intersection pt of p_i and p_j



$$\phi(\mathbb{p}) = (p_1,\dots,p_i',\dots,p_j',\dots p_n).$$

Note.

$\operatorname{sgn}(\mathbb{P}) = \operatorname{sgn}(\sigma)$

$\operatorname{sgn}(\phi(\mathbb{P})) = \operatorname{sgn}(\sigma \underbrace{(i,j)}_{\text{trans.}})$

$= -\operatorname{sgn}(\sigma)$

$\Rightarrow$ sign-reversing.

$w(\phi(\mathbb{P})) = w(\mathbb{P})$ Yes.

($\because$ set of edges used is preserved)

ϕ: involution

$\phi(\phi(\mathbb{P})) = \mathbb{P}$. $\square$

Cor. G: directed graph, edge weight w.

$\mathbb{A} = (A_1, \ldots, A_n)$, $\mathbb{B} = (B_1, \ldots, B_n)$.

$M = (M_{ij})$ $\quad M_{ij} = \sum_{p \in P(A_i \to B_j)} w(p)$.

Suppose every nonintersecting n-paths

$\mathbb{P} = (p_1, \ldots, p_n) \in P(\mathbb{A} \to \mathbb{B})$

satisfies $p_i \in P(A_i \to B_i)$. $\forall i$.

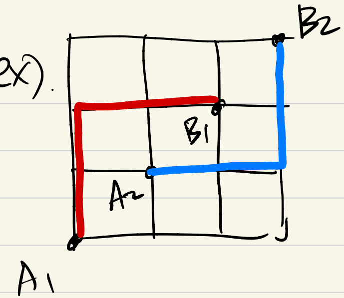
$\Rightarrow \det M = \sum_{\mathbb{P} \in NI(\mathbb{A} \to \mathbb{B})} w(\mathbb{P})$.

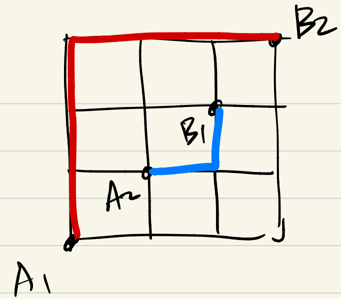
In particular, if $w(e) = 1$,

$\det M = |NI(\mathbb{A} \to \mathbb{B})|$

ex).

$w=1$

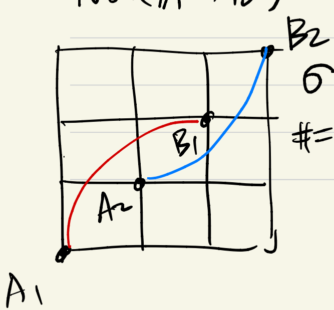




$$\det M = \det\begin{pmatrix} \binom{4}{2} & \binom{6}{3} \\ \binom{2}{1} & \binom{4}{2} \end{pmatrix}$$

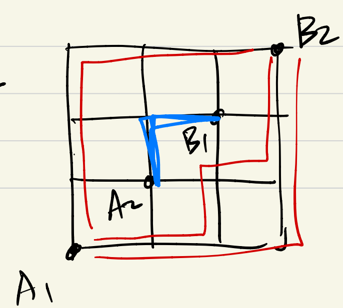
$$= \det\begin{pmatrix} 6 & 20 \\ 2 & 6 \end{pmatrix} = 36-40 = -4.$$

$NI(\mathbb{A} \to \mathbb{B})$



$\sigma = 12$

$\# = 2$



$\sigma = 21$

$\# = 6$

RHS of LGV $= \mathrm{sgn}(12)\cdot 2$
$\quad + \mathrm{sgn}(21)\cdot 6$
$\quad = 2-6 = -4.$

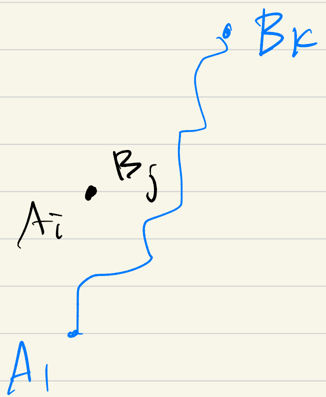
Rem What if $A_i = A_j$ (or $B_i = B_j$).

$$\det M = \sum_{p \in NI} w(p)\,\mathrm{sgn}(p) = 0$$

↳ also seen to be 0 since row i = row j.

Rem What if $A_i = B_j$?

$$P(A_i \to B_j) = \{ (A_i) \}$$



In this case

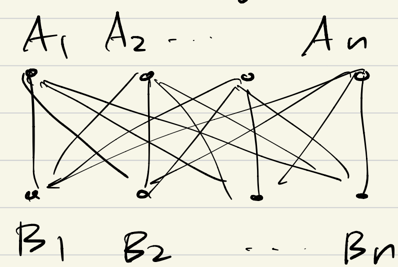
all other paths in $\mathbb{P} \in NI(\mathbb{A} \to \mathbb{B})$

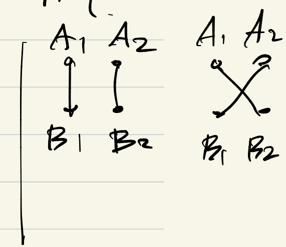
must avoid A_i.

ex) G : directed graph

$$V = \{ A_1, \ldots, A_n, B_1, \ldots, B_n \}.$$

$$E = \{ (A_i, B_j) : 1 \le i, j \le n \}.$$





$$P(A_i \to B_j) = \{ (A_i, B_j) \}.$$

$$M = (M_{ij}), \quad M_{ij} = w(A_i, B_j)$$

$$\det M = \sum_{\mathbb{P} \in NI(\mathbb{A} \to \mathbb{B})} \operatorname{sgn}(\mathbb{P})\, w(\mathbb{P}).$$

Every $\mathbb{P} \in P(\mathbb{A} \to \mathbb{B})$ is nonintersecting!

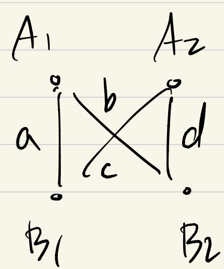
$$\det M = \sum_{\mathbb{P} \in NI(\mathbb{A} \to \mathbb{B})} \operatorname{sgn}(\mathbb{P})\, w(\mathbb{P}).$$

$$= \sum_{\mathbb{P} \in P(\mathbb{A} \to \mathbb{B})} \operatorname{sgn}(\mathbb{P})\, w(\mathbb{P}).$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} w(A_i, B_{\sigma(i)})$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) M_{i\sigma(i)}$$

let's say $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$



Def). $M = (M_{ij})_{i \in [m],\, j \in [n]}$.

let $\binom{[m]}{k}$ = set of all subsets of $[m]$ with cardinality k.

For $I \in \binom{[m]}{k}$, $J \in \binom{[n]}{k}$

the <u>(I,J)-minor</u> of M is

$$[M]_{I,J} = \det (M_{ij})_{i \in I,\, j \in J}.$$

ex) $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$[M]_{\{1,3\},\{2,3\}} = \det \begin{pmatrix} b & c \\ h & i \end{pmatrix}$$

Thm (Cauchy–Binet Thm).

M : $n \times l$ matrix

N : $l \times n$ matrix.

$$\Rightarrow \det(MN)$$

$$= \sum_{I \in \binom{[l]}{n}} [M]_{[n],I} \, [N]_{I,[n]}$$