**Thm** (Cauchy-Binet Thm).

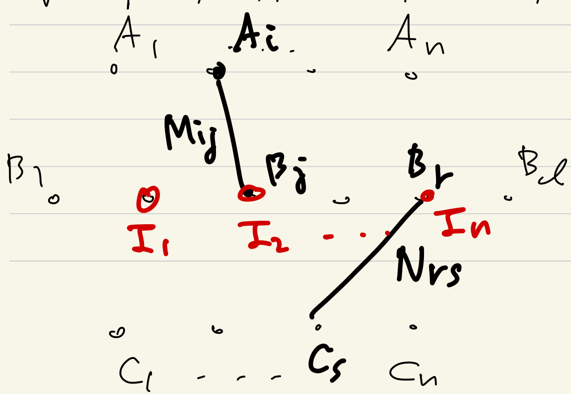
$M$ : $n \times l$ matrix

$N$ : $l \times n$ matrix.

$$\Rightarrow \det(MN) = \sum_{I \in \binom{[l]}{n}} [M]_{[n],I} [N]_{I,[n]}$$

Pf) let $G$ be the directed graph

$V = \{A_1, \dots, A_n, B_1, \dots, B_l, C_1, \dots C_n\}$




$w(A_i, B_j) = M_{ij}$, $w(B_i, C_j) = N_{ij}$

let $L = (L_{ij})$ be given by

$$L_{ij} = \sum_{p \in P(A_i \to C_j)} w(p)$$

$$= \sum_{k=1}^{l} w(A_i, B_k)\, w(B_k, C_j)$$

$$= \sum_{k=1}^{l} M_{ik} N_{kj} = (MN)_{ij}.$$

let $A = (A_1, \dots, A_n)$

$C = (C_1, \dots, C_n)$.

By LGV lem,

$$\det L = \sum_{P \in NI(A \to B)} \operatorname{sgn}(P)\, w(P)$$

Every $\mathbb{p} = (p_1, \dots p_n) \in NI(A \to B)$ is of this form

$$p_i = (A_i, B_{I_{\tau(i)}}, C_{\sigma(i)}).$$

$$I = (I_1 < \dots < I_n) \in \binom{[\ell]}{n}$$

$$\sigma = \rho\tau \iff \rho = \sigma\tau^{-1}$$

$\Rightarrow \operatorname{sgn}(p) = \operatorname{sgn}(\sigma)$.

$$\det L = \sum_{I \in \binom{[\ell]}{n}} \sum_{\tau \in S_n} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} w(A_i, B_{I_{\tau(i)}})\, w(B_{I_{\tau(i)}}, C_{\sigma(i)})$$

$$= \sum_I \sum_\tau \sum_\rho \operatorname{sgn}(\rho\tau) \prod_{i=1}^{n} w(A_i, B_{I_{\tau(i)}})\, w(B_{I_{\tau(i)}}, C_{\rho\tau(i)})$$

$$= \sum_I \underbrace{\sum_\tau \operatorname{sgn}(\tau) \prod_{i=1}^{n} \underbrace{w(A_i, B_{I_{\tau(i)}})}_{M_{i, I_{\tau(i)}}}}_{= [M]_{[n], I}} \underbrace{\sum_\rho \operatorname{sgn}(\rho) \prod_{i=1}^{n} \underbrace{w(B_{I_{\cancel{\tau(i)}\, i}}, C_{\sigma \cancel{\tau(i)}\, i})}_{N_{I_i, \sigma(i)}}}_{[N]_{I, [n]}} \qquad \square$$

## §7.3. Hankel determinants of moments.

Let $\{p_n(x)\}_{n\geq 0}$ be a monic OPS with moments $\mu_n$ s.t.

$$p_{n+1} = (x-b_n)p_n - \lambda_n p_{n-1}.$$

Def) The **Hankel matrix** $H$ of $\{\mu_n\}_{n\geq 0}$ is

$$H = (\mu_{i+j})_{i,j=0}^{\infty}$$

$$= \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots \\ \mu_1 & \mu_2 & & \\ \mu_2 & & \ddots & \\ \vdots & & & \end{pmatrix}$$

The **Hankel determinant**

$$\Delta_n = [H]_{\{0,\dots,n\},\{0,\dots,n\}}$$

$$= \det(\mu_{i+j})_{i,j=0}^{n}$$

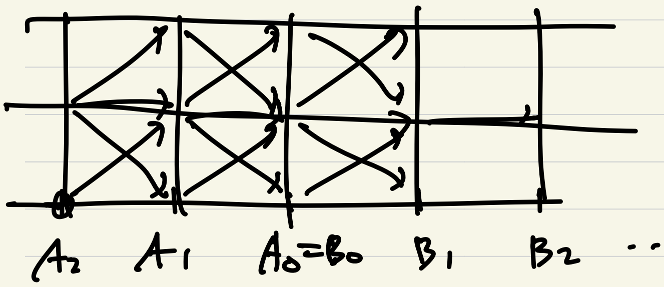
$$= \det\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & & & \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}$$

$$\mu_{i+j} = \sum_{p\in \mathrm{Motz}((0,0)\to(i+j,0))} \mathrm{wt}(p)$$

$A = (A_0, \ldots, A_n)$

$B = (B_0, \ldots, B_n)$.

$A_i = (-i, 0)$, $\quad B_i = (i, 0)$

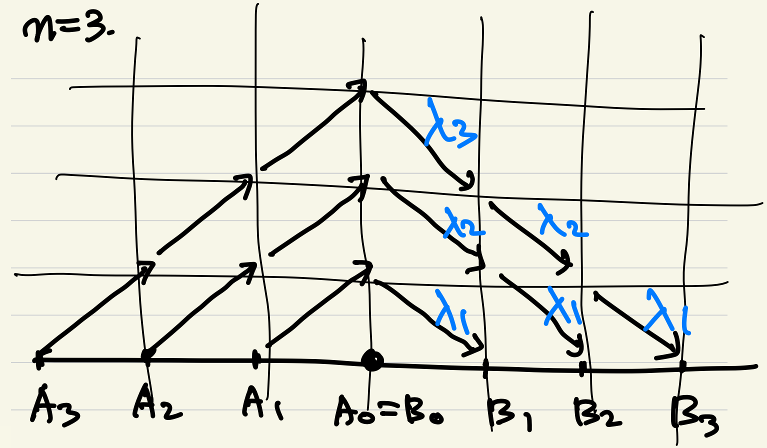



$$\mu_{i+j} = \sum_{p \in \mathrm{Motz}(A_i \to B_j)} \mathrm{wt}(p).$$

By LGV,

$$\Delta_n = \det(\mu_{ij})_0^n = \sum_{\mathbb{P} \in NI(A \to B)} \mathrm{sgn}(\mathbb{P})\, \mathrm{wt}(\mathbb{P}).$$

$n = 3$.




$NI(A \to B)$ has a unique elt. $\mathbb{P}$

$\hookrightarrow$ $\mathrm{sgn} = 1$, $\mathrm{wt} = \lambda_1^n \lambda_2^{n-1} \cdots \lambda_n^1$

<u>Thm</u> $\quad \Delta_n = \lambda_1^n \lambda_2^{n-1} \cdots \lambda_n^1$

Define

$$\Delta_n' = [H]_{\{0,\dots,n\},\{0,\dots,n-1,n+1\}}$$

$$= \det(M_{ij})_{i,j=0}^{n}.$$

$M_{ij} = \mu_{i+j}$ if $j<n$

$M_{in} = \mu_{i+n+1}.$

$A = (A_0, \dots A_n)$

$B' = (B_0, \dots B_n)$

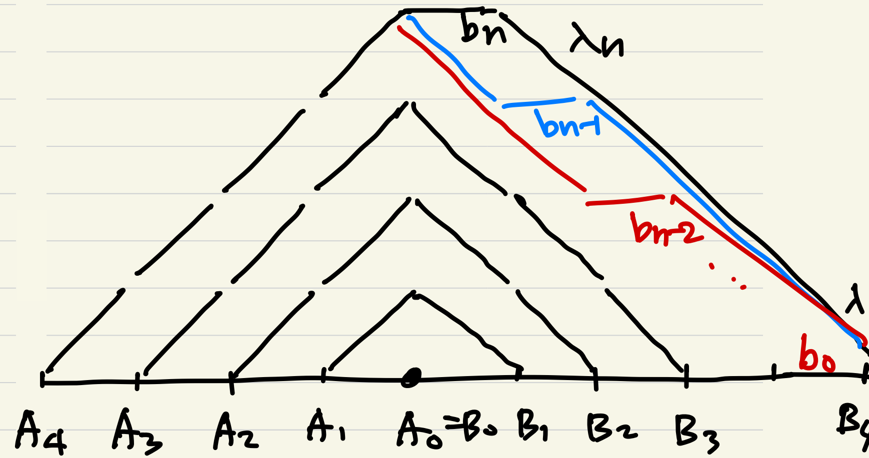
$A_i = (-i, 0).$

$B_i = (i+\delta_{n,i}, 0).$

$$\Rightarrow \Delta_n' = \det(M_{ij})_0^n$$

By LGV,

$$\Delta_n' = \sum_{P \in NI(A\to B')} \operatorname{sgn}(p)\, wt(p).$$




There are $n+1$ elts in $NI(A\to B')$ with $\operatorname{sgn}=1$ wts $=\Delta_n b_k$ $0\le k\le n$.

**Thm** $\Delta_n' = \Delta_n (b_0 + \cdots + b_n)$.

$$\left( \Delta_n = \lambda_1^n \lambda_2^{n-1} \cdots \lambda_n^1 \right)$$

$$\lambda_1 \cdots \lambda_n = \frac{\Delta_n}{\Delta_{n-1}}$$

$$\Rightarrow \quad \lambda_n = \frac{\Delta_n}{\Delta_{n-1}} \left( \frac{\Delta_{n-1}}{\Delta_{n-2}} \right)^{-1}$$

$$= \frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}$$

$$b_0 + \cdots + b_n = \frac{\Delta_n'}{\Delta_n}$$

$$b_n = \frac{\Delta_n'}{\Delta_n} - \frac{\Delta_{n-1}'}{\Delta_{n-1}}.$$

**Cor** $\lambda_n = \dfrac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}$

$$b_n = \frac{\Delta_n'}{\Delta_n} - \frac{\Delta_{n-1}'}{\Delta_{n-1}}.$$

**Cor** $\{\mu_n\}$ : a seq of numbers.

$\exists$ OPS with moments $\mu_n$

$\Longleftrightarrow \Delta_n \neq 0 \quad \forall n \geq 0$.

pf) Recall $\{P_n\}$ OPS $\Longleftrightarrow$ $\lambda_n \neq 0$.

($\Rightarrow$) $P_n$ OPS $\Rightarrow \lambda_n \neq 0 \Rightarrow \Delta_n \neq 0$.

($\Leftarrow$) Define $\lambda_n, b_n$ using Cor.

Define $P_n$ using $\lambda_n, b_n$.

$\Rightarrow$ moment of $P_n = \mu_n$. $\square$.

Now suppose $b_n = 0$. $\forall n$.

<u>Recall</u> $b_n = 0 \iff \mu_{2n+1} = 0$.

$$\Delta_3 = \det \begin{pmatrix} \boxed{\mu_0} & & \boxed{\mu_2} & \\ & \mu_2 & & \mu_4 \\ \boxed{\mu_2} & & \boxed{\mu_4} & \\ & \mu_4 & & \mu_6 \end{pmatrix}$$

$$= \det \begin{pmatrix} \mu_0 & \mu_2 \\ \mu_2 & \mu_4 \end{pmatrix} \cdot \det \begin{pmatrix} \mu_2 & \mu_4 \\ \mu_4 & \mu_6 \end{pmatrix}.$$

Let $\Delta_n(2) = \det\left(\mu_{2i+2j}\right)_{i,j=0}^{n}$

$$\Delta_n^{+}(2) = \det\left(\mu_{2i+2j+2}\right)_{i,j=0}^{n}$$

$$\Delta_4 = \det \begin{pmatrix} \boxed{\mu_0} & 0 & \boxed{\mu_2} & 0 & \boxed{\mu_4} \\ 0 & \boxed{\mu_2} & 0 & \boxed{\mu_4} & 0 \\ \boxed{\mu_2} & 0 & \boxed{\mu_4} & 0 & \boxed{\mu_6} \\ 0 & \boxed{\mu_4} & 0 & \boxed{\mu_6} & 0 \\ \boxed{\mu_4} & 0 & \boxed{\mu_6} & 0 & \boxed{\mu_8} \end{pmatrix}$$

$$= \det \begin{pmatrix} \mu_0 & \mu_2 & \mu_4 \\ \mu_2 & \mu_4 & \mu_6 \\ \mu_4 & \mu_6 & \mu_8 \end{pmatrix} \det \begin{pmatrix} \mu_2 & \mu_4 \\ \mu_4 & \mu_6 \end{pmatrix}$$

In general

$$\Delta_{2n} = \Delta_n(2)\, \Delta_{n-1}^{+}(2)$$

$$\Delta_{2n+1} = \Delta_n(2)\, \Delta_n^{+}(2).$$

Since $b_n = 0$,

$$\mu_{2n} = \sum_{p \in Dyck_{2n}} wt(p)$$

let $A = (A_0, \ldots A_n)$
$B = (B_0, \ldots B_n)$

$A_i = (-2i, 0)$, $B_i = (2i, 0)$.

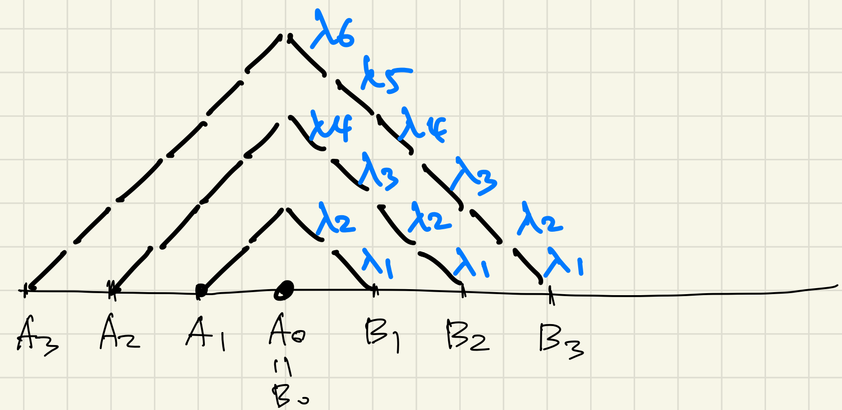
$$\mu_{2i+2j} = \sum_{p \in Dyck(A_i \to B_j)} wt(p)$$

By LGV.

$$\Delta_n(2) = \det(\mu_{2i+2j})_0^n$$

$$\Delta_n(2) = \sum_{p \in NI(A \to B)} sgn(p)\, wt(p).$$

$$= (\lambda_1 \lambda_2)^n (\lambda_3 \lambda_4)^{n-1} \cdots (\lambda_{2n-1} \lambda_{2n})^1$$



Since $b_n = 0$,

$$\mu_{2n} = \sum_{p \in \mathrm{Dyck}_{2n}} wt(p)$$

let $A = (A_0, \dots, A_n)$

$B^+ = (B_0^+, \dots, B_n^+)$

$A_i = (-2i, 0)$, $B_i^+ = (2i + 2, 0)$

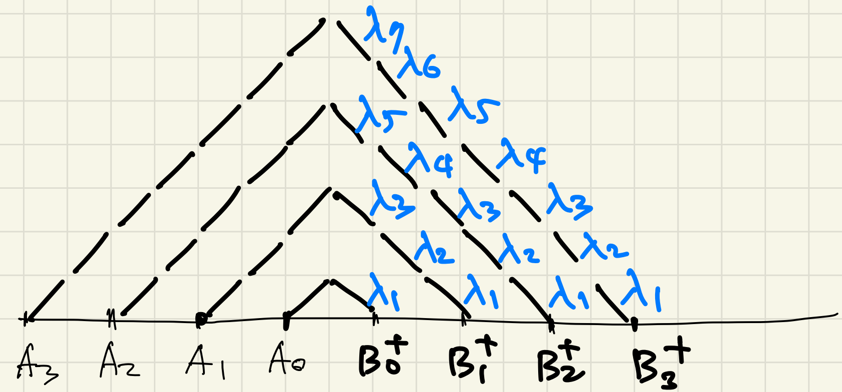
$$\mu_{2i+2j+2} = \sum_{p \in \mathrm{Dyck}(A_i \to B_j^+)} wt(p)$$

By LGV.

$$\Delta_n^+(2) = \det\left(\mu_{2i+2j+2}\right)_0^n$$

$$\Delta_n^+(2) = \sum_{p \in NI(A \to B^+)} \mathrm{sgn}(p)\, wt(p).$$

$$= \lambda_1^{n+1} (\lambda_2\lambda_3)^n (\lambda_4\lambda_5)^{n-1} \cdots (\lambda_{2n}\lambda_{2n+1})^1$$



Cor If $b_n = 0$ for all $n \geq 0$

then

$$\lambda_{2n} = \frac{\Delta_n(2)\,\Delta^{+}_{n-2}(2)}{\Delta_{n-1}(2)\,\Delta^{+}_{n-1}(2)}, \quad \lambda_{2n+1} = \frac{\Delta^{+}_{n}(2)\,\Delta_{n-1}(2)}{\Delta_n(2)\,\Delta^{+}_{n-1}(2)}.$$

pf)

$$\frac{\Delta^{+}_{n}(2)}{\Delta_n(2)} = \lambda_1 \lambda_3 \cdots \lambda_{2n+1}. \qquad \frac{\Delta_n(2)}{\Delta^{+}_{n-1}(2)} = \lambda_2 \lambda_4 \cdots \lambda_{2n}.$$