§7.4. Another duality between moments and coefficients

<u>Recall</u> We found a duality between $\mu_{n,k}$, $\nu_{n,k}$

<u>Thm</u> $\mathcal{L}$: lin ftnl with moments $\{\mu_n\}$. $\Delta_n \neq 0$.

$\Rightarrow$ The monic OPS $\{P_n(x)\}$ for $\mathcal{L}$ is given by

$$P_n(x) = \frac{1}{\Delta_{n-1}} \det \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & & & \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{pmatrix}$$

Let $P_n(x) = \sum_{k=0}^{n} \nu_{n,k} x^k$.

<u>Thm (restated)</u>

$$\nu_{n,k} = \frac{(-1)^{n-k}}{\Delta_{n-1}} [H]_{\{0,\dots,n-1\},\{0,\dots,\hat{k},\dots,n\}}$$

missing

$$[H]_{\{0,\dots,n-1\},\{0,\dots,\hat{k},\dots,n\}}$$

$$= \det\left(\mu_{i+j+\chi(j\geq k)}\right)_{i,j=0}^{n-1}$$

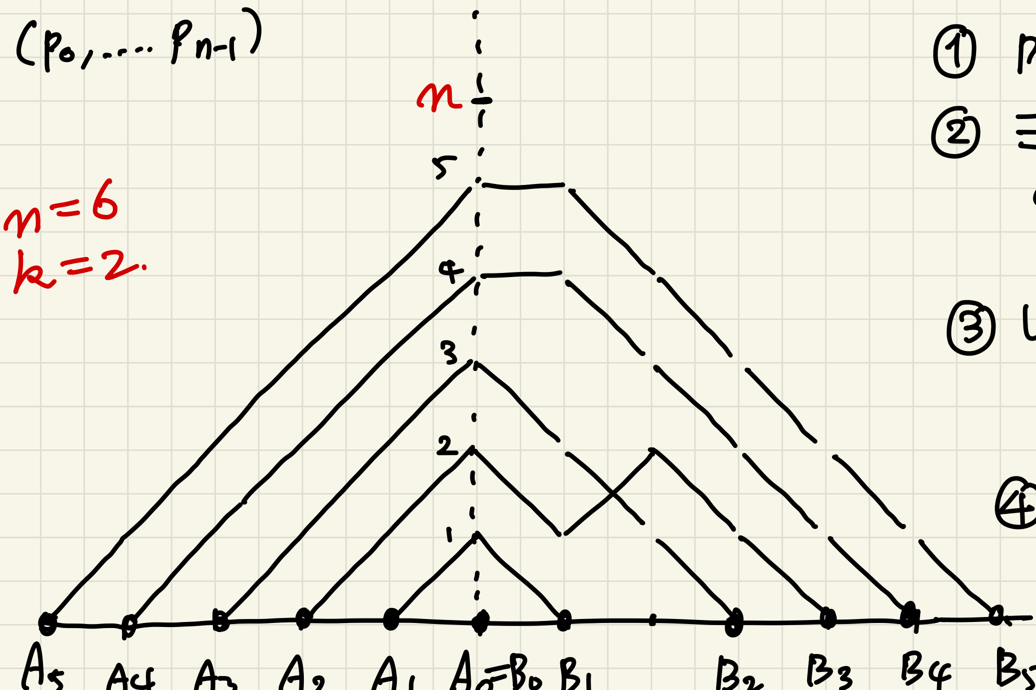
$$\left(\chi(P) = \begin{cases} 1 & \text{if } P \text{ true} \\ 0 & \text{if } P \text{ false} \end{cases} \right)$$

Let $A = (A_0, \dots, A_{n-1})$
$B^{(k)} = (B_0, \dots, B_{n-1})$

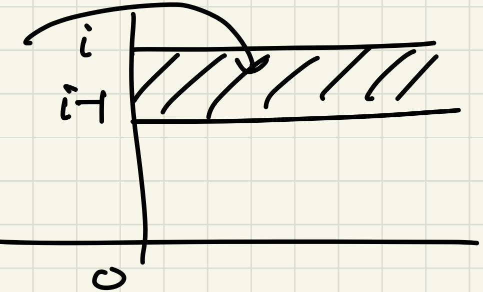
$$A_i = (-i, 0), \quad B_j = (j + \chi(j\geq k), 0).$$

$$[H]_{\{0,\dots,n-1\},\{0,\dots,\hat{k},\dots n\}} = \sum_{\substack{\mathbb{P} \in NI(\mathbb{A} \to \mathbb{B}^{(k)}) \\ \parallel \\ (p_0,\dots p_{n-1})}} \mathrm{sgn}(\mathbb{P})\, \mathrm{wt}(\mathbb{P}).$$

$n = 6$
$k = 2.$



R_i



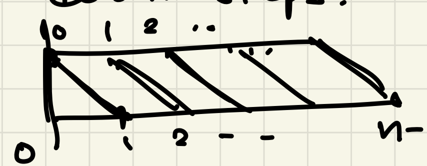
$(i \geq 1)$

① p_i has i up steps at begin.

② $\exists$ $n-i$ paths having at least one step in R_i
$(p_i, \dots\dots p_{n-i})$

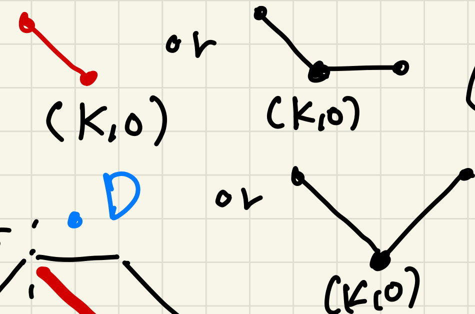
③ let $d = \#$ downs in R_i
$u = \#$ ups in R_i
$\Rightarrow d - u = n - i$

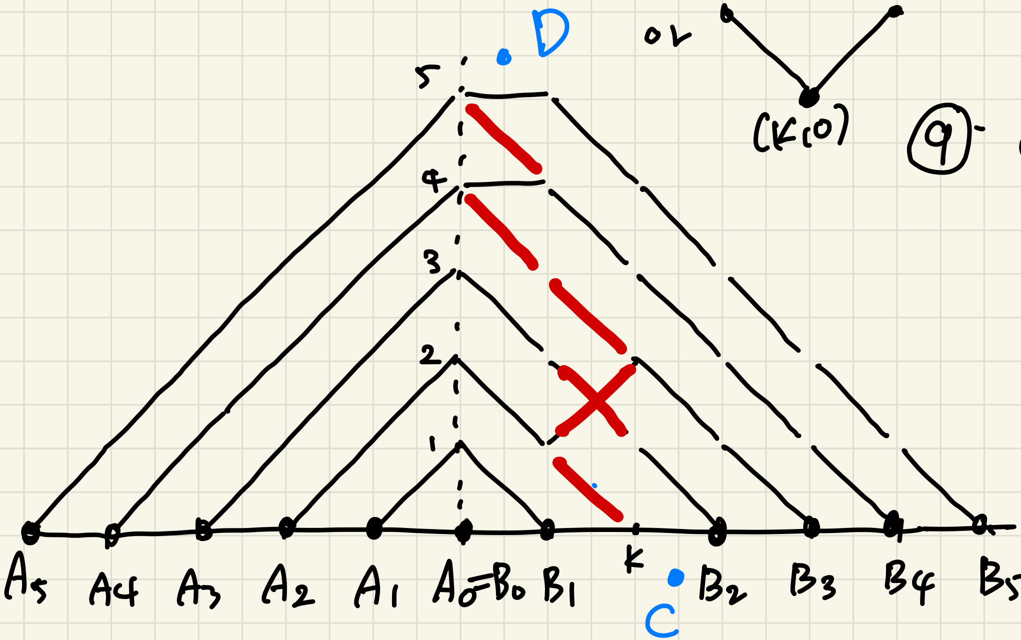
④ In R_i $\exists$ $n-i+1$ possible down steps.



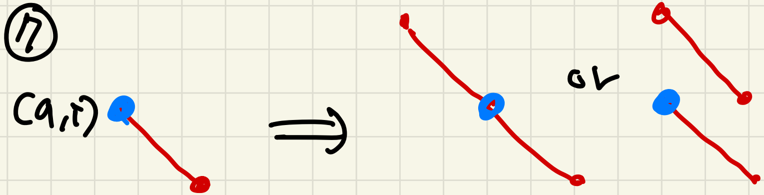
⑤ In R_i, $\exists$ unique "missing" down step or unique X but not both.

⑥ In R_1,

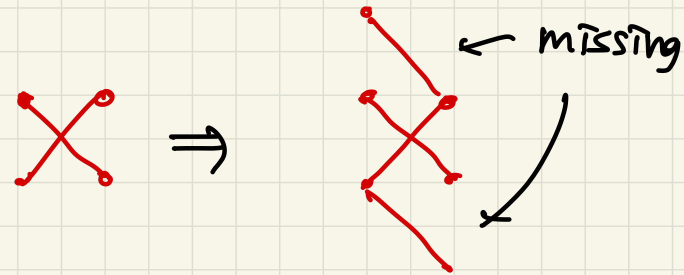




⑦



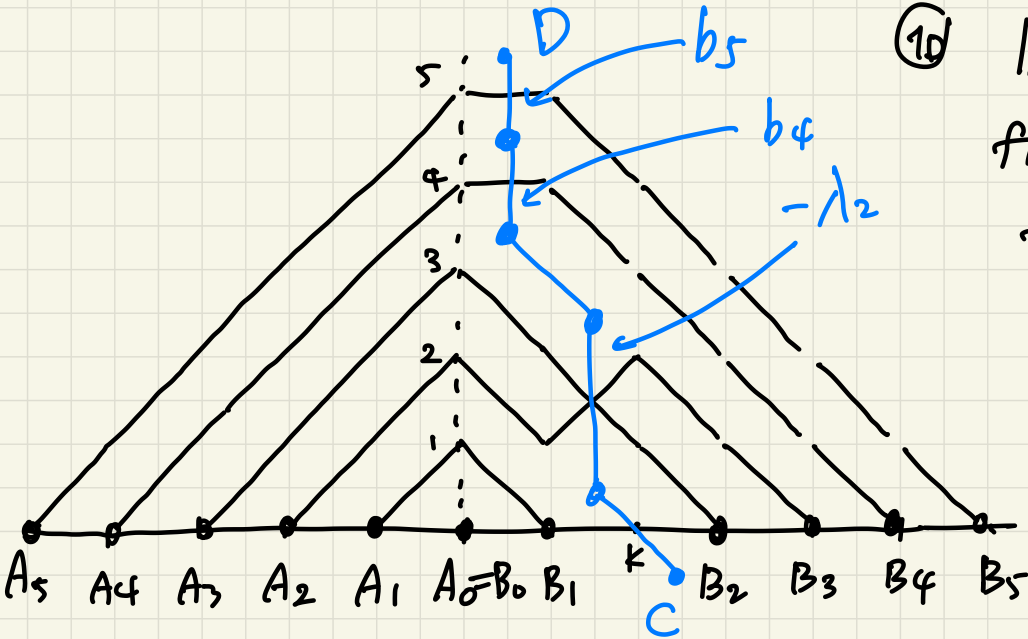
⑧



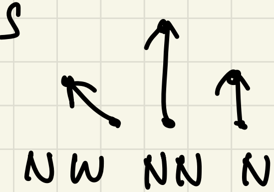
⑨ $C = (k+\frac{1}{2}, -\frac{1}{2})$

$D = (\frac{1}{2}, n+\frac{1}{2})$

Construct $p' : C \to D$ by connecting mid points of missing steps.



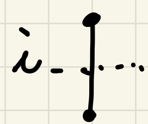
1d) $\mathbb{P} \mapsto p'$ is a bijection from $NI(A \to B^{(k)})$ to $\{$paths $p' : C \to D\}$ using steps



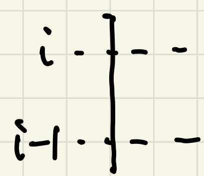
$\Rightarrow \mathrm{sgn}(\mathbb{P})\,\mathrm{wt}(\mathbb{P}) = w(p')\,\Delta_{n-1}.$

Lem. suppose $\mathbb{P} \mapsto p'$

Let $\omega(p')$ = prod of b_i



& $-\lambda_i$



__Lem.__ $\displaystyle\sum_{p':C\to D} w(p') = (-1)^{n-k}\, \nu_{n,k}.$

Pf). $p' \xleftrightarrow{1-1} T \in FT_{n,k}.$

Suppose p' has r

and s

$k = \#$

$\Rightarrow r + 2s + k = n.$

sgn of $p' = (-1)^s$

sgn of $T = (-1)^{r+s}$

sign difference

$(-1)^r = (-1)^{n-2s-k} = (-1)^{n-k}.$

Since $\displaystyle\sum_{T\in FT_{n,k}} wt(T) = \nu_{n,k}$

we get lem. $\square$.

__Pf of Thm__

$$\nu_{n,k} = (-1)^{n-k} \sum_{p':C\to D} w(p')$$

$$= (-1)^{n-k} \sum_{\mathbb{P}\in NI(\mathbb{A}\to\mathbb{B}^{(k)})} \frac{\mathrm{sgn}(\mathbb{P})\,wt(\mathbb{P})}{\Delta_{n-1}}$$

$$= \frac{(-1)^{n-k}}{\Delta_{n-1}} [H]_{\{0,\dots,n-1\},\{0,\dots,\hat{k},\dots,n\}}.$$

Lem (Inverse minor formula)

$A = (A_{i,j})_{i,j=0}^{n}$

$I, J \subseteq \{0, \ldots, n\}$ with $|I| = |J|$.

$$\Rightarrow [A^{-1}]_{I,J} = (-1)^{\|I\| + \|J\|} \frac{[A]_{J',I'}}{\det A}$$

$\|I\| = \sum_{i \in I} i$, $I' = \{0, \ldots n\} \setminus I$.

(If $|I| = |J| = 1$, this is cofactor formula for A^{-1}.)

Let $A = (\mu_{i+j})_{i,j=0}^{n}$.

$\det A = \Delta_n = \Delta_{n-1} \lambda_1 \cdots \lambda_n$

$$\nu_{n,k} = \frac{(-1)^{n+k} \lambda_1 \cdots \lambda_n}{\det A} [A]_{\{0,\ldots,n\}\setminus\{n\},\ \{0,\ldots,n\}\setminus\{k\}}$$

$J = \{0,\ldots,n\}\setminus\{n\}$, $I = \{0,\ldots,n\}\setminus\{k\}$

$$= \lambda_1 \cdots \lambda_n (A^{-1})_{k,n}$$

$$\Rightarrow (\lambda_1 \cdots \lambda_n)^{-1} \nu_{n,k} = (A^{-1})_{k,n}$$

$\Rightarrow (\lambda_1 \cdots \lambda_n)^{-1} \nu_{n,k}$ are the entries of last col of A^{-1}.

$$\Leftrightarrow \sum_{k=0}^{n} \mu_{i+k} v_{n,k} = \delta_{n,i} \lambda_1 \cdots \lambda_n$$

$$\Leftrightarrow \mathcal{L}(x^i P_n(x)) = \delta_{n,i} \lambda_1 \cdots \lambda_n.$$

So, our thm is basically a formula for $(A^{-1})_{k,n}$.

Q: $(A^{-1})_{r,s} = ?$

<u>Thm</u> let $A = (\mu_{i+j})_{i,j=0}^{n}$

$$(A^{-1})_{r,s} = \sum_{k=0}^{n} \frac{v_{k,r} v_{k,s}}{\lambda_1 \cdots \lambda_k}$$

Pf) $I = \{r\}$, $J = \{s\}$.

$$(A^{-1})_{r,s} = (-1)^{r+s} \frac{[A]_{J',I'}}{\det A}.$$

$A^{(s)} = (A_0, \ldots, A_{n-1})$

$B^{(r)} = (B_0, \ldots, B_{n-1})$.

$A_i = (-i - \chi(i \geq s), 0)$

$B_i = (i + \chi(i \geq r), 0)$

By LGV,

$$[A]_{J',I'} = \sum_{\mathbf{p} \in NI(\mathbb{A}^{(s)} \to \mathbb{B}^{(r)})} \operatorname{sgn}(\mathbf{p}) \operatorname{wt}(\mathbf{p}).$$

$$[A]_{J',I'} = \sum_{\mathbb{P} \in NI(\mathbb{A}^{(s)} \to \mathbb{B}^{(r)})} \operatorname{sgn}(\mathbb{P}) \operatorname{wt}(\mathbb{P}).$$

$$= \sum_{k=\max(r,s)}^{n} \frac{\Delta_n}{\lambda_1 \cdots \lambda_k} (-1)^{k-r} \nu_{k,r} (-1)^{k-s} \nu_{k,s} \qquad \square$$

<u>Pf2</u> using orthogonality

$$\delta_{r,s} = \sum_{k=0}^{n} \mu_{r+k} (A^{-1})_{k,s} = \sum_{k=0}^{n} \mu_{r+k} \sum_{l=0}^{n} \frac{\nu_{l,k} \nu_{l,s}}{\lambda_1 \cdots \lambda_l}.$$

$$\text{RHS} = \mathcal{L}\left(\sum_{k=0}^{n} x^{r+k} \sum_{l=0}^{n} \frac{\nu_{l,k}\nu_{l,s}}{\lambda_1 \cdots \lambda_l} \right) = \mathcal{L}\left(x^r \sum_{l=0}^{n} \frac{\nu_{l,s}}{\lambda_1 \cdots \lambda_l} \sum_{k=0}^{n} \nu_{l,k} x^k \right)$$

$$= \mathcal{L}\left(\sum_{a=0}^{r} \mu_{r,a} P_a(x) \sum_{l=0}^{n} \frac{\nu_{l,s}}{\lambda_1 \cdots \lambda_l} P_l(x) \right).$$

$$= \sum_{l=0}^{n} \mu_{r,l} \nu_{l,s} = \delta_{r,s}.$$

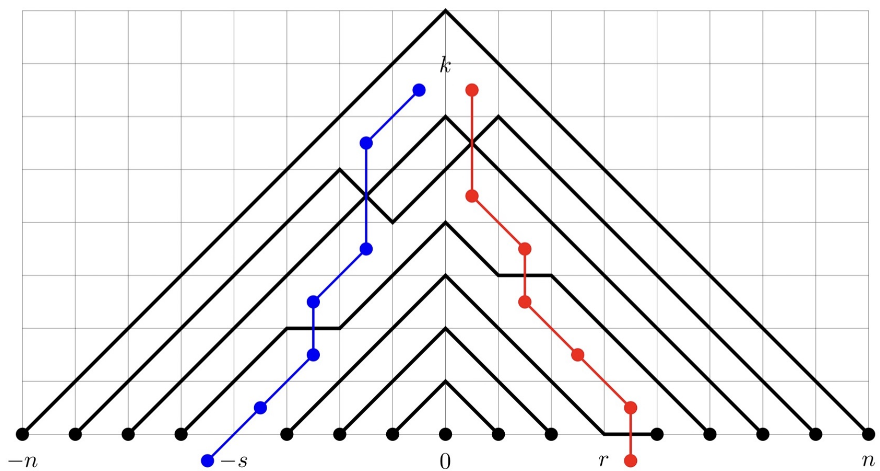


Figure 7.12: A nonintersecting $(n+1)$-path in $\mathrm{NI}(\mathbf{A}^{(s)} \to \mathbf{B}^{(r)})$. The red (resp. blue) path on the left contributes to $(-1)^{k-s}\nu_{k,s}$ (resp. $(-1)^{k-r}\nu_{k,r}$).

Proof 2. We can also prove this theorem using orthogonality. Given, r, s, it suffces to prove the validity of