Thm Let $A=(\mu_{i+j})_{i,j=0}^{n}$

$$(A^{-1})_{r,s} = \sum_{k=0}^{n} \frac{\nu_{k,r}\,\nu_{k,s}}{\lambda_1\cdots\lambda_k}$$

$$(\nu_{i,j})^{-1} = (\mu_{i,j})$$

UDL-decomp

$$\left((\mu_{i+j})_{i,j=0}^{n}\right)^{-1} = \left((\nu_{i,j})_{i,j=0}^{n}\right)^{T} D(\lambda_1^{-1},\dots,\lambda_1^{-1}\cdots\lambda_n^{-1})\,(\nu_{i,j})_{i,j=0}^{n}.$$

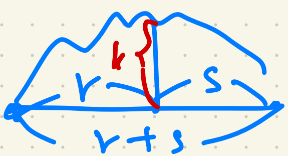
$$\Rightarrow (\mu_{i+j})_{i,j=0}^{n} = (\mu_{i,j})_{i,j=0}^{n}\, D(\lambda_1,\dots,\lambda_1\cdots\lambda_n)\left((\mu_{i,j})_{i,j=0}^{n}\right)^{T}$$

LDU-decomp.

$$\Rightarrow \mu_{r+s} = \sum_{k=0}^{n} \mu_{r,k}\,\lambda_1\cdots\lambda_k\,\mu_{s,k}$$

$$= \sum_{k=0}^{n} \mu_{r,0,k}\,\lambda_1\cdots\lambda_k\,\mu_{s,0,k}$$

$$= \sum_{k=0}^{n} \mu_{r,0,k}\,\mu_{s,k,0}$$



Ch 8. Continued Fractions

§8.1. Basics of continued fr.

ex)
$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} = x$$

$$\frac{1}{1+x} = x$$

$$x^2 + x - 1 = 0$$

$$x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{\sqrt{5}-1}{2}$$

ex)
$$\cfrac{1}{1-\cfrac{3}{1-\cfrac{3}{1-\cdots}}} = x$$

$$\frac{1}{1-3x} = x$$

$$3x^2 - x + 1 = 0$$

$$x = \frac{1 \pm \sqrt{1-12}}{2}$$

$$= \frac{1 \pm \sqrt{11}\, i}{2}$$

Def). A <u>continued fraction</u> is an expression of the form

$$\beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \cfrac{\alpha_3}{\ddots}}}$$

for some sequences $\{\beta_n\}$, $\{\alpha_n\}$.

The <u>nth convergent</u> is

$$C_n = \beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \cdots + \cfrac{\alpha_n}{\beta_n}}}$$

e.g. $C_0 = \beta_0$

$$C_1 = \beta_0 + \frac{\alpha_1}{\beta_1}$$

$$C_2 = \beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2}}$$

We write

$$\beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \cdots}} = L$$

to mean

$$L = \lim_{n\to\infty} C_n.$$

§8.2. Flajolet's combinatorial theory of continued fractions.

$$\mathbb{b} = (b_0, b_1, \ldots)$$

$$\lambda = (\lambda_1, \lambda_2, \ldots)$$

Define

$$\mu_n(\mathbb{b}, \lambda) = \sum_{\pi \in \mathrm{Motz}_n} \mathrm{wt}(\pi; \mathbb{b}, \lambda)$$

let

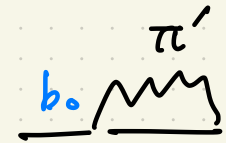
$$F(x; \mathbb{b}, \lambda) = \sum_{n \geq 0} \mu_n(\mathbb{b}, \lambda) x^n$$

Observation $\pi \in \mathrm{Motz}$.

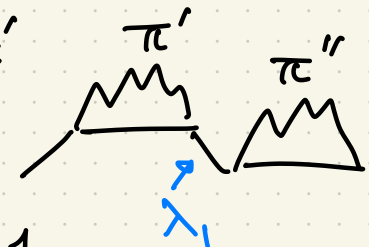
There are 3 cases.

① $\pi = \phi$.

② $\pi = H\pi'$



③ $\pi = U\pi' D\pi''$

$\pi =$ 

① : $\mathrm{wt}(\pi; \mathbb{b}, \lambda) = 1$

② : $\mathrm{wt}(\pi; \mathbb{b}, \lambda) = b_0 \mathrm{wt}(\pi'; \mathbb{b}, \lambda)$

③ : $\mathrm{wt}(\pi; \mathbb{b}, \lambda)$

$$= \lambda_1 \mathrm{wt}(\pi'; \delta\mathbb{b}, \delta\lambda)\, \mathrm{wt}(\pi''; \mathbb{b}, \lambda)$$

$$\delta\mathbb{b} = (b_1, b_2, b_3 \ldots)$$

$$\delta\lambda = (\lambda_2, \lambda_3, \ldots)$$

$$F(x;\mathbb{b},\boldsymbol{\lambda}) = \sum_{\pi} wt(\pi;\mathbb{b},\boldsymbol{\lambda})$$

$$= 1 + b_0 x \sum_{\pi'} wt(\pi';\mathbb{b},\boldsymbol{\lambda})$$

$$+ \lambda_1 x^2 \sum_{\pi'} wt(\pi';\delta\mathbb{b},\delta\boldsymbol{\lambda}) \sum_{\pi''} wt(\pi'';\mathbb{b},\boldsymbol{\lambda}).$$

$$= 1 + b_0 x F(x;\mathbb{b},\boldsymbol{\lambda}) + \lambda_1 x^2 F(x;\delta\mathbb{b},\delta\boldsymbol{\lambda}) F(x;\mathbb{b},\boldsymbol{\lambda})$$

$$\Rightarrow \quad F(x;\mathbb{b},\boldsymbol{\lambda}) = \frac{1}{1 - b_0 x - \lambda_1 x^2 F(x;\delta\mathbb{b},\delta\boldsymbol{\lambda})}$$

$$= \cfrac{1}{1 - b_0 x - \cfrac{\lambda_1 x^2}{1 - b_1 x - \lambda_2 x^2 F(x;\delta^2\mathbb{b},\delta^2\boldsymbol{\lambda})}}$$

$$\Rightarrow \quad F(x; \mathbb{b}, \mathbb{\lambda}) = \cfrac{1}{1 - b_0 x - \cfrac{\lambda_1 x^2}{1 - b_1 x - \cfrac{\lambda_2 x^2}{1 - b_2 x - \cfrac{\lambda_3 x^2}{\ddots}}}}$$

Def). $\{F_n(x)\}_{n \geq 0}$: a seq of formal power series.

$\lim\limits_{n \to \infty} F_n(x) = F(x) \iff$ $\forall m \geq 0 \ \exists N > 0$ such that for all $n > N$, $[x^m] F_n(x) = \underbrace{[x^m] F(x)}_{\text{coeff of } x^m}$

ex) $F(x) = 1 + x + x^2 + \cdots$

$F_n(x) = 1 + x + \cdots + x^n$

$$\lim_{n \to \infty} F_n(x) \overset{?}{=} F(x)$$

x^m

Def) $\mu_n^{\le k}(\mathbb{b}, \lambda) = \sum_{\pi \in \mathrm{Motz}_n^{\le k}} \mathrm{wt}(\pi; \mathbb{b}, \lambda)$

$\mathrm{Motz}_n^{\le k} =$  k

$\mathrm{ht} \le k.$

<u>Thm</u> For any $k \ge 0$,

$$\sum_{n\ge 0} \mu_n^{\le k}(\mathbb{b}, \lambda) x^n = \cfrac{1}{1 - b_0 x - \cfrac{\lambda_1 x^2}{1 - b_1 x - \cfrac{\lambda_2 x^2}{\ddots - \cfrac{\lambda_k x^2}{1 - b_k x}}}}$$

Pf) Ind on k. $k=0$. $\sum_{n\ge 0} \mu_n^{\le 0}(\mathbb{b}, \lambda) x^n = \sum_{n\ge 0} b_0^n x^n = \frac{1}{1-b_0 x}$

Let $k \geq 1$. Suppose true for $k-1$.
For the case k, consider decomp of π.

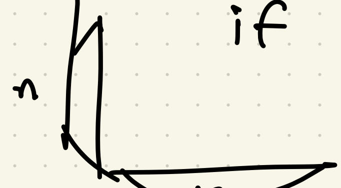
$\pi = \bullet$ or  or 

Let $F_k(x; \mathbb{b}, \boldsymbol{\lambda}) = \sum_{n \geq 0} \mu_n^{\leq k}(\mathbb{b}, \boldsymbol{\lambda}) x^n$

$$F_k(x; \mathbb{b}, \boldsymbol{\lambda}) = 1 + b_0 x F_k(x; \mathbb{b}, \boldsymbol{\lambda}) + \lambda_1 x^2 F_{k-1}(x; \delta\mathbb{b}, \delta\boldsymbol{\lambda}) \cdot F_k(x; \mathbb{b}, \boldsymbol{\lambda})$$

$$\Rightarrow F_k(x; \mathbb{b}, \boldsymbol{\lambda}) = \frac{1}{1 - b_0 x - \lambda_1 x^2 F_{k-1}(x; \delta\mathbb{b}, \delta\boldsymbol{\lambda})}$$

$\Rightarrow$ true for k.



<u>Cor</u> $\sum_{n \geq 0} \mu_n(\mathbb{b}, \boldsymbol{\lambda}) x^n = F$

$$= \cfrac{1}{1 - b_0 x - \cfrac{\lambda_1 x^2}{1 - b_1 x - \cdots}}$$

Pf) By thm, k-th convergent

$$F_k = \sum_{n \geq 0} \mu_n^{\leq k} x^n$$

$$\lim_{k \to \infty} F_k \overset{?}{=} F$$

For fixed n,

$$\mu_n^{\leq k} = \mu_n$$

if k is big enough. $\square$

§8.3. Continued fractions and orthogonal polynomials.

let $\{p_n(x;\mathbb{b},\lambda)\}_{n\geq 0}$ be the monic OPS such that

$$p_{n+1}(x) = (x-b_n)p_n(x) - \lambda_n p_{n-1}(x)$$

let $\delta p_n(x;\mathbb{b},\lambda) = p_n(x;\delta\mathbb{b},\delta\lambda)$.

Def) The <u>inverted polynomial</u>

$$p_n^*(x;\mathbb{b},\lambda) = x^n p_n(x^{-1};\mathbb{b},\lambda).$$

$$\Rightarrow p_{n+1}^*(x) = (1-b_n x)p_n^*(x) - \lambda_n x^2 p_{n-1}^*(x).$$

<u>Thm</u>

$$\sum_{n\geq 0} \mu_n^{\leq k}(\mathbb{b},\lambda)\, x^n = \frac{\delta p_k^*(x;\mathbb{b},\lambda)}{p_{k+1}^*(x;\mathbb{b},\lambda)}.$$

We will use a technique due to John Wallis (1616-1703).

$\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 0}$.

$$C_n = \beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \ddots + \cfrac{\alpha_n}{\beta_n}}}$$

$$C_n = \frac{A_n}{B_n} \ ?$$

$$C_0 = \frac{\beta_0}{1} = \frac{A_0}{B_0}$$

$$C_1 = \beta_0 + \frac{\alpha_1}{\beta_1} = \frac{\beta_1\beta_0 + \alpha_1}{\beta_1} = \frac{A_1}{B_1}$$

$$\vdots$$

Lem (Wallis)

Let $A_0 = \beta_0$, $B_0 = 1$.

$A_{-1} = 1$, $B_{-1} = 0$.

$$A_n = \beta_n A_{n-1} + \alpha_n A_{n-2} \qquad (n \geq 1)$$

$$B_n = \beta_n B_{n-1} + \alpha_n B_{n-2}$$

$$\Rightarrow \frac{A_n}{B_n} = C_n.$$

pf) Ind on n.

$n=0$. True!, $n=1$ True!

Suppose true for n.

$$C_{n+1} = \beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \cdots + \cfrac{\alpha_n}{\beta_n + \cfrac{\alpha_{n+1}}{\beta_{n+1}}}}}$$

$$= \beta_0 + \cfrac{\alpha_1}{\beta_1 + \cfrac{\alpha_2}{\beta_2 + \cdots \cfrac{\alpha_n \beta_{n+1}}{\beta_n \beta_{n+1} + \alpha_{n+1}}}}$$

$$\Rightarrow C_{n+1} = \frac{A_n^*}{B_n^*}$$

$A_n^* = A_n$ with $\alpha_n \mapsto \alpha_n \beta_{n+1}$

$B_n^* = B_n$ 〃 $\beta_n \mapsto \beta_n \beta_{n+1} + \alpha_{n+1}$

$$A_n^* = (\beta_n \beta_{n+1} + \alpha_{n+1}) A_{n-1} + \alpha_n \beta_{n+1} A_{n-2}$$

$$= \beta_{n+1} (\beta_n A_{n-1} + \alpha_n A_{n-2}) + \alpha_{n+1} A_{n-1}$$

$$= \beta_{n+1} A_n + \alpha_{n+1} A_{n-1} = A_{n+1}$$

$$\Rightarrow B_n^* = B_{n+1}$$

$$\Rightarrow C_{n+1} = \frac{A_{n+1}}{B_{n+1}} \qquad \square$$

Pf of thm $\displaystyle\sum_{n\geq 0} \mu_n^{\leq k}(\mathbb{b},\lambda)\, x^n = \frac{\delta p_k^*(x;\mathbb{b},\lambda)}{p_{k+1}^*(x;\mathbb{b},\lambda)}.$

$p_{n+1}^*(x) = (1-b_n x)\, p_n^*(x) - \lambda_n x^2 p_{n-1}^*(x).$

$$\sum_{n\geq 0} \mu_n^{\leq k}(\mathbb{b},\lambda)\, x^n = \cfrac{1}{1-b_0x-\cfrac{\lambda_1x^2}{1-b_1x-\cfrac{\lambda_2x^2}{\ddots - \cfrac{\lambda_kx^2}{1-b_kx}}}}$$

Use Lem with

$$\beta_i = 1-b_ix,\quad \alpha_i = -\lambda_i x^2$$

$$\Rightarrow \frac{A_k}{B_k} = 1-b_0x-\cfrac{\lambda_1x^2}{1-b_1x-\cfrac{\lambda_2x^2}{\ddots - \cfrac{\lambda_kx^2}{1-b_kx}}}$$

$$A_{-1}=1,\quad A_0 = 1-b_0x$$

$$A_n = (1-b_nx)A_{n-1} - \lambda_n x^2 A_{n-2}$$

$$\Rightarrow \quad A_n = p_{n+1}^*(x),\quad B_n =$$

$$B_n = (1-b_nx)B_{n-1} - \lambda_n x^2 B_{n-2}.$$

$$B_{-1}=0,\quad B_1 = 1.$$

$$\Rightarrow B_n = \delta p_n^*(x)$$

$$\Rightarrow C_k = \frac{A_k}{B_k} = \frac{p_{k+1}^*(x)}{\delta p_k^*(x)}$$