* Sign-Reversing Involutions

Def) A <u>sign</u> of a set X is a function $\mathrm{sgn}: X \to \{+1, -1\}$. A <u>sign-reversing involution</u> on X is an involution $\phi: X \to X$ such that

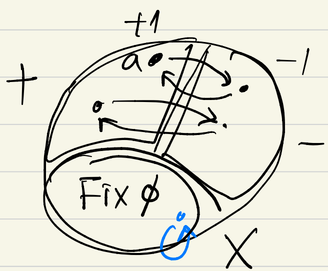
① $\mathrm{sgn}(\phi(x)) = -\mathrm{sgn}(x)$ for all $x \in X \setminus \mathrm{Fix}(\phi)$,

② $\mathrm{sgn}(x) = 1$ for all $x \in \mathrm{Fix}(\phi)$.

where $\mathrm{Fix}(\phi) = \{x \in X : \phi(x) = x\}$.

If ϕ is a sign-reversing involution on X,

$$\sum_{x \in X} \mathrm{sgn}(x) = |\mathrm{Fix}(\phi)|.$$

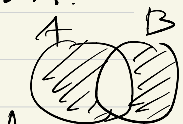


ex) $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$

pf) let X be the set of all subsets of $[n]$. ($[n] = \{1, 2, \ldots, n\}$.)
For $A \in X$, define $\mathrm{sgn}(A) = (-1)^{|A|}$.
We want to find a s.r.i. on X with no fixed pts.

$$\hookrightarrow \sum_{A \in X} \mathrm{sgn}(A) = \sum_{A \in X} (-1)^{|A|} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}.$$

For $A \in X$, define $\phi(A) = A \triangle \{1\}$.
$\left(A \triangle B = (A \cup B) - (A \cap B) \right)$



$$\phi(A) = A \triangle \{1\} = \begin{cases} A \setminus \{1\} & \text{if } 1 \in A \\ A \cup \{1\} & \text{if } 1 \notin A \end{cases}$$

$\Rightarrow$ ϕ is a s.r.i. on X with no fixed points.

ex) ⊛ $\cdots \displaystyle\sum_{k\geq 0} P_m(k)P_n(k)\frac{a^k}{k!} = \frac{e^a a^n}{n!}\delta_{n,m}$

$$P_n(x) = \sum_{k=0}^{n}\binom{x}{k}\frac{(-a)^{n-k}}{(n-k)!}$$

Pf) LHS of ⊛

$$= \sum_{k\geq 0}\sum_{i=0}^{m}\binom{k}{i}\frac{(-a)^{m-i}}{(m-i)!}\sum_{j=0}^{n}\binom{k}{j}\frac{(-a)^{n-j}}{(n-j)!}\frac{a^k}{k!}$$

$$= \sum_{k\geq 0}\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{k}{m-i}\frac{(-a)^i}{i!}\binom{k}{n-j}\frac{(-a)^j}{j!}\frac{a^k}{k!}$$

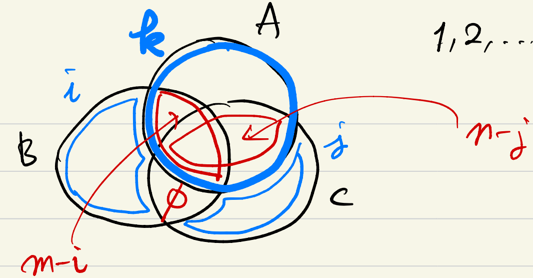
$$= \sum_{N\geq 0}\frac{a^N}{N!}\sum_{i+j+k=N}(-1)^{i+j}\frac{N!}{i!j!k!}\binom{k}{m-i}\binom{k}{n-j}$$

Here, $\binom{r}{s}=0$ if $s<0$.

Fix N.

(**) $\cdots \displaystyle\sum_{i+j+k=N}(-1)^{i+j}\binom{N}{i,j,k}\binom{k}{m-i}\binom{k}{n-j} = \sum_{(A,B,C)\in X}(-1)^{|B\setminus A|+|C\setminus A|}$

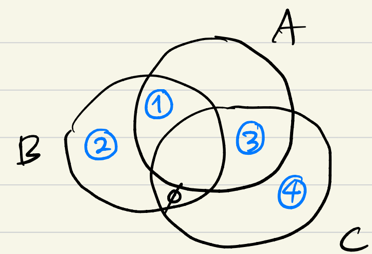
X = set of triples (A,B,C) s.t. $A\cup B\cup C=[N]$
$|A|=k,\ |B|=m,\ |C|=n,\ (B\cap C)\setminus A=\phi$.



$1,2,\ldots,N$

For $(A,B,C)\in X$, define $\mathrm{sgn}(A,B,C)=(-1)^{|B\setminus A|+|C\setminus A|}$

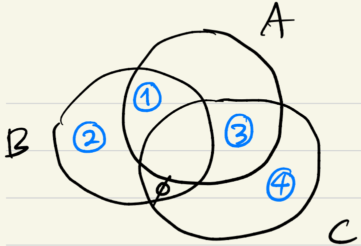
LHS of (**) $= \displaystyle\sum_{(A,B,C)\in X}\mathrm{sgn}(A,B,C)$.



For $(A,B,C)\in X$, define $\phi(A,B,C)$ as follows.

case I: All regions ① ② ③ ④ are empty.
$\phi(A,B,C)=(A,B,C)$.

case II. At least of of ①, ②, ③, ④ is nonempty.
Let s be the smallest elt in ①, ② ③ ④.



$$\operatorname{sgn}(A,B,C) = (-1)^{|B\setminus A|+|C\setminus A|}$$

For $(A,B,C)\in X$, define $\phi(A,B,C)$ as follows.

case I: All regions ①, ②, ③, ④ are empty.

$$\phi(A,B,C) = (A,B,C).$$

case II. At least of of ①, ②, ③, ④ is nonempty.

Let s be the smallest elt in ①, ②, ③, ④.

If $s\in$ ① : move s to ②
② → ①
③ → ④
④ → ③.

Let (A',B',C') be the resulting sets.

Define $\phi(A,B,C) = (A',B',C')$.

$$\operatorname{sgn}(A,B,C) = (-1)^{|B\setminus A|+|C\setminus A|}$$
$$= -(-1)^{|B'\setminus A'|+|C'\setminus A'|}$$
$$= -\operatorname{sgn}(A',B',C').$$

Thus ϕ is a sign-reversing involution on X.

$$\operatorname{Fix}\phi = \{(A,B,C)\in X : B=C\subseteq A\}.$$

$$\sum_{(A,B,C)\in X} \operatorname{sgn}(A,B,C) = |\operatorname{Fix}\phi| = \binom{N}{n}\delta_{m,n}$$

If $B=C\subseteq A$, then $A\cup B\cup C = A = [N]$.

This can happen only if $\underset{|B|}{\overset{\shortparallel}{m}} = \underset{|C|}{\overset{\shortparallel}{n}}$.

If $m=n$, $|\operatorname{Fix}\phi| = \binom{N}{n}$. $= \frac{N!}{n!(N-n)!}$

Therefore,

$$\sum_{k\geq 0} P_m(k)P_n(k)\frac{a^k}{k!} = \sum_{N\geq 0}\frac{a^N}{N!}\delta_{m,n}\binom{N}{n}$$
$$= \delta_{m,n}\cdot\frac{e^a a^n}{n!}$$