Let $P(x), Q(x)$ be poly of deg p, q.

We say $\dfrac{P(x)}{Q(x)}$ is a <u>Padé approximant</u> of type (p,q) for $F(x)$ if

$$F(x) - \frac{P(x)}{Q(x)} = \sum_{n \geq p+q+1} a_n x^n .$$

<u>Fact</u> If $F(x)$ is a power series with a Padé approx. of type (p,q), then it is unique.

<u>Cor</u> Let $F(x) = \sum_{n \geq 0} \mu_n(\mathbf{b}, \boldsymbol{\lambda}) x^n$.

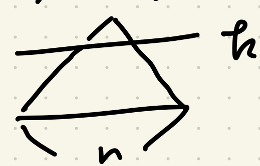
For $k \geq 0$, $\dfrac{\delta^* p_k^*(x; \mathbf{b}, \boldsymbol{\lambda})}{p_{k+1}^*(x; \mathbf{b}, \boldsymbol{\lambda})}$ is the Padé approximant of $F(x)$ of type $(k, k+1)$.

pf).

$$R(x) = \frac{\delta^* p_k^*(x; \mathbf{b}, \boldsymbol{\lambda})}{p_{k+1}^*(x; \mathbf{b}, \boldsymbol{\lambda})} = \sum_{n \geq 0} \mu_n^{\leq k} x^n .$$

$$F(x) - R(x) = \sum_{n \geq 0} \left(\mu_n - \mu_n^{\leq k}\right) x^n$$

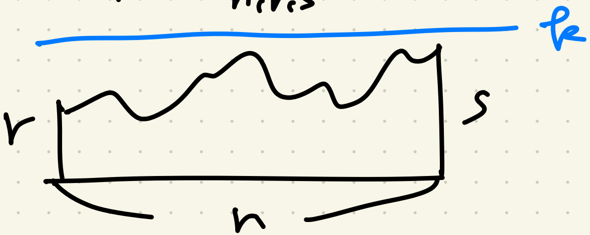
Suppose $\mu_n \neq \mu_n^{\leq k}$.

$\Rightarrow$ 

$\Rightarrow n \geq (k+1)2 \geq k+(k+1)+1$. □

§8.4. Motzkin paths with fixed starting and ending heights.

Def) $\mathrm{Motz}_{n,r,s} = \mathrm{Motz}((0,r)\to(n,s))$



$$\mathrm{Motz}^{\le k}_{n,r,s} = \{\pi \in \mathrm{Motz}_{n,r,s} \mid \max \mathrm{ht} \le k\}.$$

$$\mu^{\le k}_{n,r,s} = \sum_{\pi \in \mathrm{Motz}^{\le k}_{n,r,s}} \mathrm{wt}(\pi; b, \lambda)$$

Thm Let $0 \le r, s \le k$.

$$\sum_{n\ge 0} \mu^{\le k}_{n,r,s}\, x^k = \begin{cases} x^{s-r}\, \dfrac{p^*_r(x)\, \delta^{s+1} p^*_{k-s}(x)}{p^*_{k+1}(x)} & \text{if } r \le s \\[2ex] \lambda_{s+1}\cdots\lambda_r\, x^{r-s}\, \dfrac{p^*_s(x)\, \delta^{r+1} p^*_{k-r}(x)}{p^*_{k+1}(x)} & \text{if } r > s. \end{cases}$$

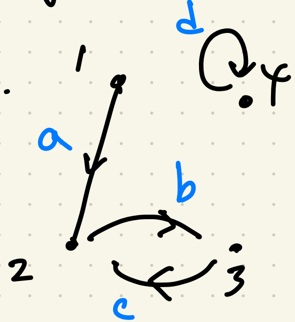
$G=(V,E)$: directed graph.

$V=[m]$. , $w: E \to K$

$A=(a_{ij})_{i,j=1}^{m}$: adjacency matrix of weighted graph G.

$a_{ij}=w(i\to j)$.

ex).



$$A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \end{array}$$

Def). A path of len n from u to v is $p=(u_0,u_1,\dots,u_n)$ such that

$u_0=u$, $u_n=v$

$u_{i-1}\to u_i \in E$

$$w(p)=w(u_0\to u_1)\cdots w(u_{n-1}\to u_n)$$

$$=a_{u_0,u_1}\cdots a_{u_{n-1},u_n}.$$

$P_n(u\to v)=\{$all paths of length n from u to $v\}$

$$P(u\to v)=\bigcup_{n\geq 0} P_n(u\to v).$$

Lem. Let $i,j \in [m]$. $n \geq 0$.

$$\sum_{p \in P_n(i \to j)} w(p) = (A^n)_{i,j}$$

Pf) Immediate from def of A^n.

Prop Suppose $I-A$ invertible.

$$\sum_{p \in P(i \to j)} w(p) = \left((I-A)^{-1}\right)_{i,j}$$

$$= \frac{(-1)^{i+j} [I-A]_{\{j\}^c, \{i\}^c}}{\det(I-A)}.$$

Pf) $$\sum_{p \in P(i \to j)} w(p) = \sum_{n \geq 0} \sum_{p \in P_n(i \to j)} w(p)$$

$$= \sum_{n \geq 0} (A^n)_{ij} = \left(\sum_{n \geq 0} A^n\right)_{ij}$$

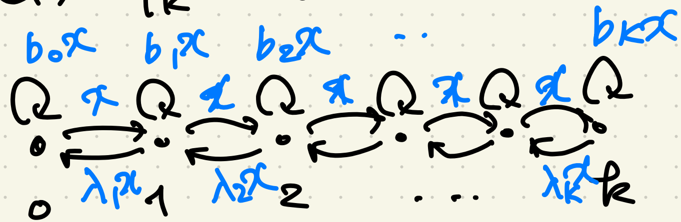
$$= \left((I-A)^{-1}\right)_{ij}.$$

$\square$

$$A = \begin{bmatrix} b_0 x & x & & \\ \lambda_1 x & \ddots & \ddots & \\ & \ddots & \ddots & x \\ & & \lambda_k x & b_k x \end{bmatrix}$$

$$\sum_{n \geq 0} \mu^{\leq k}_{n,r,s} x^n = \sum_{p \in P_n(r \to s)} \omega(p)$$

Def) $G_k = (V, E)$.

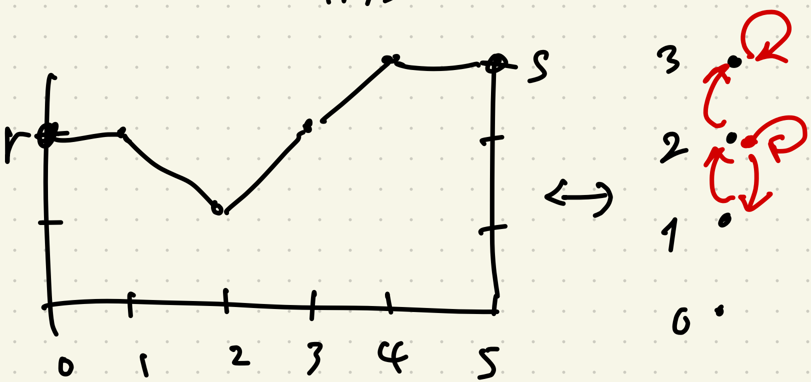


By prop,

$$\sum_{n \geq 0} \mu^{\leq k}_{n,r,s} x^n = \frac{(-1)^{r+s} [I - A]_{\{s\}^c, \{r\}^c}}{\det(I - A)}.$$

$\pi \in \mathrm{Motz}^{\leq k}_{n,r,s} \longleftrightarrow p \in P_n(r \to s).$

$$I - A = \begin{bmatrix} 1 - b_0 x & -x & & \\ -\lambda_1 x & 1 - b_1 x & \ddots & \\ & \ddots & \ddots & -x \\ & & -\lambda_k x & 1 - b_k x \end{bmatrix}$$



$$\sum_{n\geq 0} \mu_{n,r,s}^{\leq k} x^n = \frac{(-1)^{r+s}[I-A]_{\{s\}^c,\{r\}^c}}{\det(I-A)}.$$

$$I-A = \begin{bmatrix} 1-b_0x & -x & & \\ -\lambda_1 x & 1-b_1x & \ddots & \\ & \ddots & \ddots & -x \\ & & -\lambda_k x & 1-b_k x \end{bmatrix}$$

Lem $\det(I-A) = p^*_{k+1}(x)$

Pf) Let $Q_{k+1}(x) = \det(I-A)$.

$$\Rightarrow Q_{k+1}(x) = (1-b_k x)\,Q_k(x) - \lambda_k x^2 Q_{k-1}(x).$$

same rec as p^*

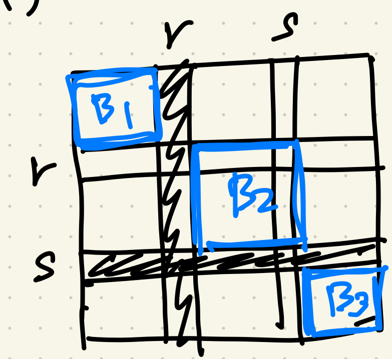
same initial cond.

$Q_0 = 1,\ Q_1 = 1-b_0X.$

Lem $(-1)^{r+s}[I-A]_{\{s\}^c,\{r\}^c}$

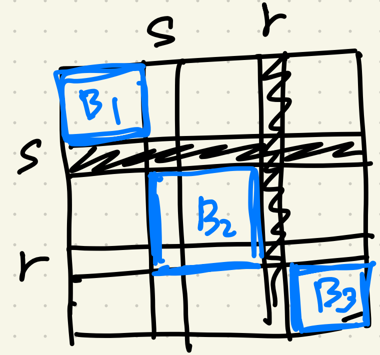
$$= \begin{cases} x^{s-r} p_r^* \,\delta^{s+1} p^*_{k-s} & \text{if } r \leq s \\ \lambda_{s+1}\cdots\lambda_r x^{r-s} p_s^* \,\delta^{r+1} p^*_{k-r} & \text{if } r > s. \end{cases}$$

Pf) $r \leq s.$



$$\text{LHS} = \underbrace{\det B_1}_{=\,p_r^*}\,\underbrace{\det B_2}_{=\,(-1)^{s-r}x^{s-r}}\,\underbrace{\det B_3}_{=\,\delta^{s+1}p^*_{k-s}}$$

If $r > s$



$$\det B_2 = (-1)^{s-r}\lambda_{s+1}\cdots\lambda_r x^{r-s}$$

□

§8.5. A combinatorial proof using disjoint paths and cycles.

Let $G=(V,E)$ directed graph.

$V=[m]$. $w: E \to K$.

$A=(a_{i,j})$: adj mat of G.

Def) A <u>cycle</u> is a path $p=(u_0,\dots,u_n)$ such that $u_0=u_n$.

We will identify p with cyclic shift.

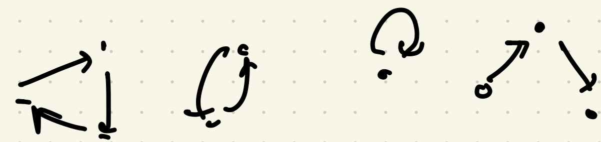
$p=(u_j,\dots,u_n,u_0,\dots u_j)$.

A path $p=(u_0,\dots u_n)$ is <u>self-avoiding</u> if $u_i \neq u_j$ for all i,j except $u_0=u_n$

A collection $\{p_1,\dots,p_t\}$ of paths is <u>disjoint</u> if

① p_i is self-avoiding

② $p_i \cap p_j$, $i \neq j$.



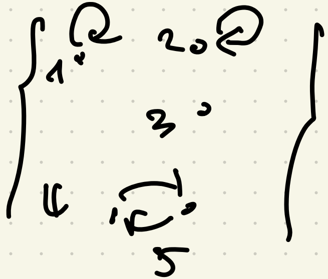
Def)

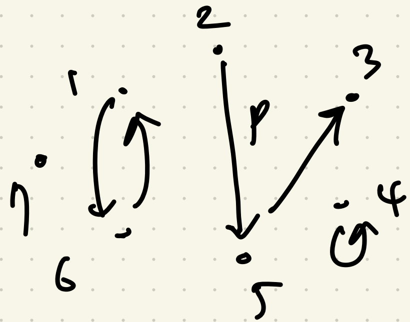
$$\Gamma = \sum_{\{C_1,\dots C_t\}\in \mathcal{C}} (-1)^t w(C_1)\cdots w(C_t)$$

$\mathcal{C}$ = set of collections of disjoint cycles in G.

$$\Gamma_{i,j} = \sum_{(p,\{C_1,\dots,C_t\})\in \mathcal{C}_{i,j}} w(p)(-1)^t w(C_1)\cdots w(C_t)$$

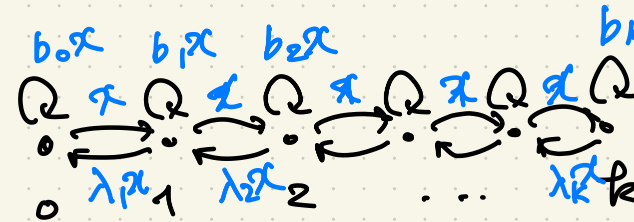
p: path, $\{p, C_1,\dots C_t\}$: disjoint.

ex) $\Gamma \supseteq$ 

$\Gamma_{2,3}$ 

Prop $$\sum_{p \in P(i,j)} w(p) = \frac{\Gamma_{i,j}}{\Gamma}$$

want to prove : $$\sum_{n \geq 0} \mu_{n,r,s}^{\leq k} x^k = \begin{cases} x^{s-r} \dfrac{p_r^*(x)\, \delta^{s+1} p_{k-s}^*(x)}{p_{k+1}^*(x)} & \text{if } r \leq s \\ \lambda_{s+1} \cdots \lambda_r x^{r-s} \dfrac{p_s^*(x)\, \delta^{r+1} p_{k-r}^*(x)}{p_{k+1}^*(x)} & \text{if } r > s. \end{cases}$$

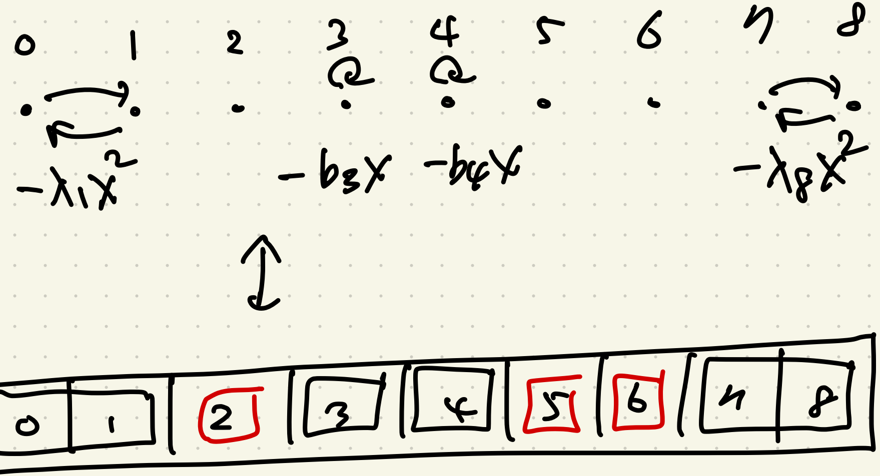
G_k : 

$$\sum_n \mu_{n,r,s}^{\leq k} x^k = \frac{\Gamma_{r,s}}{\Gamma}$$

Γ : has only cycles of leng 1 or 2.

Any $(c_1, \dots, c_t) \in \Gamma$ can be identified with $T \in FT_{k+1}$.

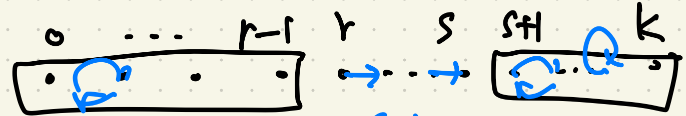
ex)



$\Rightarrow \Gamma = p^*_{k+1}(x).$

$\Gamma_{r,s} = ?$

$(r \le s)$



p^*_r $\quad x^{s-r}$ $\quad \delta^{s+1} p^*_{k-s}$

$(r > s)$

p^*_s $\quad \lambda_{s+1} \cdots \lambda_r \cdot x^{r-s}$ $\quad \delta^{r+1} p^*_{k-r}$