Prop $\displaystyle\sum_{p \in P(r\to s)} w(p) = \frac{\Gamma_{r,s}}{\Gamma}$

Pf) $\displaystyle\Gamma \sum_{p \in P(r\to s)} w(p) = \Gamma_{r,s}$.

$$\text{LHS} = \sum_{(p,\{C_1,\dots,C_t\}) \in X} w(p)\cdot(-1)^t w(C_1)\cdots w(C_t)$$

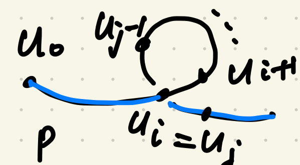
$X = \{(p,\{C_1,\dots,C_t\}) \mid p \in P(r\to s)$
$\{C_1,\dots C_t\}$: disjoint cycles $\}$

Goal: Find sign-reversing involution
$\phi : X \to X$, $\mathrm{Fix}\,\phi = \mathcal{C}_{r,s}$

Let $(p,\{C_1,\dots,C_t\}) \in X$.
If $(p,\{C_1,\dots,C_t\}) \in \mathcal{C}_{r,s}$
then $\phi(p,\{C_1,\dots,C_t\}) = (p,\{C_1,\dots C_t\})$.

Suppose $\{p, C_1,\dots,C_t\}$ not disjoint.
Let $p = (u_0, u_1, \dots u_n)$.
We can find smallest j such that
$u_i = u_j$ for some $0 \le i < j$
or $u_j \in C_\ell$ for some ℓ.

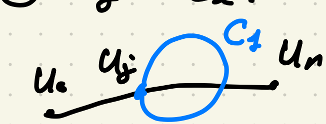
① $u_i = u_j$ for some $0 \le i < j$.



$\phi(p,\{C_1,\dots C_t\})$
$\quad \| $
$(p', \{C_0, C_1, \dots C_t\})$

$p' = (u_0, \dots u_i, u_{j+1}, \dots, u_n)$.
$C_0 = (u_i, u_{i+1}, \dots u_j)$

② $u_j \in C_\ell$. We may assume $\ell = 1$.



Let $C_1 = (v_0, v_1, \dots v_g)$.
$\quad\quad\quad\quad\quad u_j''$
$\phi(p,\{C_1,\dots C_t\})$
$= (p', \{C_2, \dots C_t\})$

$p' = (u_0, \dots, u_j, v_1, \dots, v_g, u_{j+1}, \dots, u_n)$.

§8.6. Determinants and disjoint cycles

We prove the formula for $\sum \mu_{n,r,s}^{\leq k} x^n$ in two ways. Key tools:

① $$\sum_{p \in P(r\to s)} w(p) = \frac{(-1)^{r+s} [I-A]_{\{s\}^c, \{r\}^c}}{\det(I-A)}$$

② $$\sum_{p \in P(r\to s)} w(p) = \frac{\Gamma_{r,s}}{\Gamma}$$

$G = (V, E)$: directed graph

$V = [m]$, $w : E \to K$

adj mat $A = (a_{ij})_{i,j=1}^{m}$

$$\det A = \sum_{\pi \in S_m} \operatorname{sgn}(\pi) \prod_{i=1}^{m} a_{i,\pi(i)}$$

If $\pi = C_1 \cdots C_t$ is a cycle decomp.

$\Rightarrow$ $\operatorname{sgn}(\pi) = (-1)^{\text{evencycle}(\pi)}$.

$$\operatorname{sgn}(\pi) \prod_{i=1}^{m} a_{i,\pi(i)} = \prod_{i=1}^{t} (-1)^{|C_t|-1} w(C_i)$$

$$\det A = \sum_{\{C_1,\dots,C_t\} \in X} \prod_{i=1}^{t} (-1)^{|C_t|-1} w(C_i).$$

X = set of all collections of disjoint cycles whose union (as a set) is $[m]$.

<u>Prop</u> $\det(I-A) = \Gamma$

Pf) $$\det(I-A) = \sum_{\{C_1\dots C_t\} \in X} \prod_{i=1}^{t} (-1)^{|C_t|-1} w'(C_i)$$

$w'(C) = 1 - w(C)$ if C len $= 1$.

$w'(C) = (-1)^{|C|} w(C)$ if C len > 1.

$$= \sum_{\{C_1,\dots C_t\} \in \mathcal{C}} \prod_{i=1}^{t} (-1) w(C_i)$$

$$= \Gamma.$$

$\square$

Def) For a word $w = w_1 \cdots w_n$ of distinct integers, the standardization of w is $st(w) = \pi = \pi_1 \cdots \pi_n \in S_n$ such that $w_i < w_j \iff \pi_i < \pi_j$.

ex) $w = 4\,3\,1\,7\,9$.

$\pi = 3\,2\,1\,4\,5$

Lem $\pi = \pi_1 \cdots \pi_n \in S_n$ such that $\pi_i = j$.

let $\pi' = \pi_1 \cdots \pi_{i-1} \pi_{i+1} \cdots \pi_n$

$\Rightarrow$ $sgn(st(\pi')) = (-1)^{i+j} sgn(\pi)$.

pf) $sgn(st(\pi')) = (-1)^{inv(\pi')}$

$sgn(\pi) = (-1)^{inv(\pi)}$

let $\sigma = j\,\pi'$

$(-1)^{inv(\sigma)} = (-1)^{inv(\pi') + j-1}$.

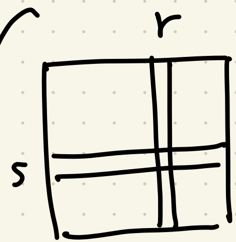
$(-1)^{inv(\sigma)} = (-1)^{inv(\pi) + i-1}$

$\Rightarrow (-1)^{inv(\pi')} = (-1)^{inv(\pi) + i+j}$ $\square$

<u>Prop</u> $(-1)^{r+s}[I-A]_{\{s\}^c,\{r\}^c} = \Gamma_{r,s}$

Pf) let $b_{ij} = \delta_{i,j} - a_{ij}$.

$$[I-A]_{\{s\}^c,\{r\}^c} = \sum_{\substack{\pi\in S_m \\ \pi(s)=r}} \operatorname{sgn}(st(\pi')) \prod_{\substack{i=1 \\ i\neq s}}^{m} b_{i,\pi(i)}$$



$\pi' = \pi_1 \cdots \pi_{s-1}\pi_{s+1}\cdots \pi_n$

$$= \sum_{\substack{\pi\in S_m \\ \pi(s)=r}} (-1)^{r+s} \operatorname{sgn}(\pi) \prod_{\substack{i=1 \\ i\neq s}}^{m} b_{i,\pi(i)}$$

<u>Case I</u> $r=s$.

$$[I-A]_{\{s\}^c,\{r\}^c} = \sum_{\substack{\pi\in S_m \\ \pi(r)=r}} \operatorname{sgn}(\pi) \prod_{\substack{i=1 \\ i\neq r}}^{m} b_{i,\pi(i)}$$

$$= \sum_{\{C_1,\dots,C_t\}\in \mathcal{C}'} \prod_{i=1}^{t} (-1)\, w(C_i)$$

$\mathcal{C}'$ = set of collections $\{C_1, \dots C_t\}$ of disjoint cycles in G such that r is not used.

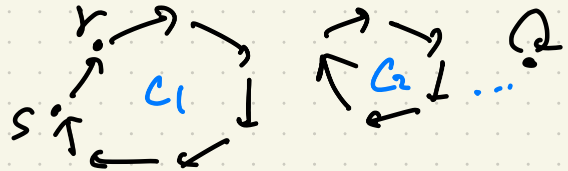
$$= \sum_{(p,\{C_1,\dots,C_t\})\in C_{r,r}} w(p)(-1)^t\, w(C_1)\cdots w(C_t)$$

$$= \Gamma_{r,r}.$$

Case II $r \neq s$. Then $b_{r,s} = -a_{r,s}$

Consider $\pi \in S_m$ with $\pi(s) = r$.

π :



$$(-1)^{r+s}[I-A]_{\{s\}^c,\{r\}^c}$$

$$= (-a_{r,s}^{-1}) \sum_{\substack{\{C_1,\dots,C_t\}\in X \\ (s\to r)\in C_1}} \prod_{i=1}^{t} (-1)^{|C_i|-1} w'(C_i).$$

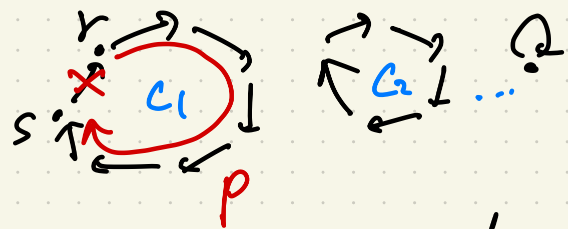
$$w'(C) = \begin{cases} 1 - w(C) & \text{if } C \text{ has leng } 1 \\ -w(C) & \text{if } \quad " \quad > 1. \end{cases}$$

By same idea,

$$(-a_{r,s}^{-1}) \sum_{\{C_1,\dots,C_t\}\in C'} \prod_{i=1}^{t} (-1) w(C_i)$$

where C' = set of collections $\{C_1,\dots,C_t\}$ of disjoint cycles in G such that $(s\to r)\in C_1$.

If we delete the edge $s\to r$ we get $(p, \{C_2,\dots C_t\}) \in C_{r,s}$



$$= \sum_{(p,\{C_1,\dots C_t\})\in C_{r,s}} w(p) \prod_{i=1}^{t} (-1) w(C_i)$$

$$= \Gamma_{r,s}.$$

$\square$.