# Ch 9. Symmetric orthogonal polynomials

## §9.1. Even and odd polynomials of symmetric OPS

<u>Recall</u> A linear functional $\mathcal{L}$ is <u>symmetric</u> if all odd moments are zero. ($\mathcal{L}(x^{2n+1}) = 0$).

Def) $\{P_n(x)\}_{n\geq 0}$ : monic OPS for $\mathcal{L}$. $\{P_n(x)\}_{n\geq 0}$ is <u>symmetric</u> if $\mathcal{L}$ is symmetric.

<u>Thm</u> $\{P_n(x)\}_{n\geq 0}$ : monic OPS for $\mathcal{L}$.

① $\mathcal{L}$ is symmetric ①′ $\{P_n(x)\}$ sym

② $P_n(-x) = (-1)^n P_n(x)$ for all $n \geq 0$.

③ $P_{n+1}(x) = x P_n(x) - \lambda_n P_{n-1}(x)$.

Suppose $\{P_n(x)\}$ is symmetric and

$$P_{n+1}(x) = x P_n(x) - \lambda_n P_{n-1}(x)$$

with lin ftn $\mathcal{L}$. ($\mathcal{L}(1) = 1$).

We have

$$P_{2n}(-x) = (-1)^{2n} P_{2n}(x) = P_{2n}(x)$$

$$P_{2n+1}(-x) = -P_{2n+1}(x).$$

$$\Rightarrow \quad P_{2n}(x) = E_n(x^2)$$

$$P_{2n+1}(x) = x\, O_n(x^2)$$

$E_n(x), O_n(x)$ : polynomials.

$\{E_n(x)\}_{n\geq 0}$ : <u>even poly</u> for $\{P_n(x)\}$

$\{O_n(x)\}_{n\geq 0}$ : <u>odd poly</u> " .

Def) linear functionals $\mathcal{L}^e, \mathcal{L}^o$:

$$\mathcal{L}^e(f(x)) = \mathcal{L}(f(x^2))$$

$$\mathcal{L}^o(f(x)) = \frac{1}{\lambda_1}\mathcal{L}(x^2 f(x^2)).$$

<u>Note</u> $\mathcal{L}^e(1) = \mathcal{L}(1) = 1$

$$\mathcal{L}^o(1) = \frac{1}{\lambda_1}\mathcal{L}(x^2) = 1.$$

Def) $\mu_n^e = \mathcal{L}^e(x^n)$, $\mu_n^o = \mathcal{L}^o(x^n)$.

<u>Thm</u> $\{E_n(x)\}_{n\geq 0}$ : OPS for $\mathcal{L}^e$

$\{O_n(x)\}_{n\geq 0}$ : OPS " $\mathcal{L}^o$.

$$\mu_n^e = \mu_{2n}$$

$$\mu_n^o = \frac{1}{\lambda_1}\mu_{2n+2}$$

pf) $\{p_n(x)\}$ : OPS for $\mathcal{L}$

$$\mathcal{L}(p_n(x)p_m(x)) = \delta_{n,m}K_n \quad (K_n \neq 0).$$

$$\mathcal{L}^e(E_n(x)E_m(x)) = \mathcal{L}(E_n(x^2)E_m(x^2))$$

$$= \mathcal{L}(P_{2n}(x)P_{2m}(x)) = \delta_{n,m}K_{2n}.$$

$\rightarrow \{E_n\}$ OPS for $\mathcal{L}^e$.

$$\mathcal{L}^o(O_n(x)O_m(x)) = \frac{1}{\lambda_1}\mathcal{L}(x^2 O_n(x^2)O_m(x^2))$$

$$= \frac{1}{\lambda_1}\mathcal{L}(P_{2n+1}(x)P_{2m+1}(x))$$

$$= \frac{1}{\lambda_1}\delta_{n,m}K_{2n+1}.$$

$\Rightarrow \{O_n\}$ OPS for $\mathcal{L}^o$

$$\mu_n^e = \mathcal{L}^e(x^n) = \mathcal{L}(x^{2n}) = \mu_{2n}$$

$$\mu_n^o = \mathcal{L}^o(x^n) = \frac{1}{\lambda_1}\mathcal{L}(x^2\cdot x^{2n}) = \frac{1}{\lambda_1}\mu_{2n+2}.$$

$\square$

Thm.

$$E_{n+1}(x) = (x - \lambda_{2n} - \lambda_{2n+1}) E_n(x) - \lambda_{2n-1}\lambda_{2n} E_{n-1}(x)$$

$$O_{n+1}(x) = (x - \lambda_{2n+1} - \lambda_{2n+2}) O_n(x) - \lambda_{2n}\lambda_{2n+1} O_{n-1}(x).$$

Pf) $x P_{2n+1}(x) = x^2 P_{2n}(x) - \lambda_{2n} x P_{2n-1}(x) \rightarrow O_n(x^2) = x^2 E_n(x^2) - \lambda_{2n} O_{n-1}(x^2)$

$P_{2n+2}(x) = x P_{2n+1}(x) - \lambda_{2n+1} P_{2n}(x). \rightarrow E_{n+1}(x^2) = O_n(x^2) - \lambda_{2n+1} E_n(x^2)$

$\Rightarrow O_n(x) = x E_n(x) - \lambda_{2n} O_{n-1}(x) \rightarrow x E_n = O_n + \lambda_{2n} O_{n-1}$

$E_{n+1}(x) = O_n(x) - \lambda_{2n+1} E_n(x) \rightarrow O_n = E_{n+1} + \lambda_{2n+1} E_n$

$\rightarrow x E_n = (E_{n+1} + \lambda_{2n+1} E_n) + \lambda_{2n}(E_n + \lambda_{2n-1} E_{n-1})$

$\rightarrow E_{n+1} = (x - \lambda_{2n} - \lambda_{2n+1}) E_n - \lambda_{2n-1}\lambda_{2n} E_{n-1}$

$\rightarrow x O_n = (O_{n+1} + \lambda_{2n+2} O_n) + \lambda_{2n+1}(O_n + \lambda_{2n} O_{n-1})$

$\rightarrow O_{n+1} = (x - \lambda_{2n+1} - \lambda_{2n+2}) O_n - \lambda_{2n}\lambda_{2n+1} O_{n-1}.$

<u>Cor</u> $\{\lambda_k\}_{k\geq 0}$ : seq with $\lambda_0 = 0$.

Define $\quad \mathbb{0} = \{b_k = 0\}_{k\geq 0}, \quad \boldsymbol{\lambda} = \{\lambda_k\}_{k\geq 1}$

$\quad \mathbb{b}^e = \{\lambda_{2k} + \lambda_{2k+1}\}_{k\geq 0}, \quad \boldsymbol{\lambda}^e = \{\lambda_{2k-1}\lambda_{2k}\}_{k\geq 1}$

$\quad \mathbb{b}^o = \{\lambda_{2k+1} + \lambda_{2k+2}\}_{k\geq 0}, \quad \boldsymbol{\lambda}^o = \{\lambda_{2k}\lambda_{2k+1}\}_{k\geq 0}$.

$\Rightarrow$

$$\mu_{2n}(\mathbb{0}, \boldsymbol{\lambda}) = \mu_n(\mathbb{b}^e, \boldsymbol{\lambda}^e)$$

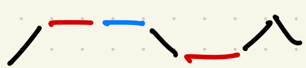
$$\mu_{2n+2}(\mathbb{0}, \boldsymbol{\lambda}) = \lambda_1 \mu_n(\mathbb{b}^o, \boldsymbol{\lambda}^o).$$

## §9.2. Converting Dyck paths into bi-colored Motzkin paths.

Def) A <u>bi-colored Motzkin path</u> is a Motzkin path in which every horizontal step is colored red or blue.

$U, D$

$H_r, H_b$



$\mathrm{Motz}_n(2)$ = set of all bi-colored Motzkin paths from $(0,0)$ to $(n,0)$.

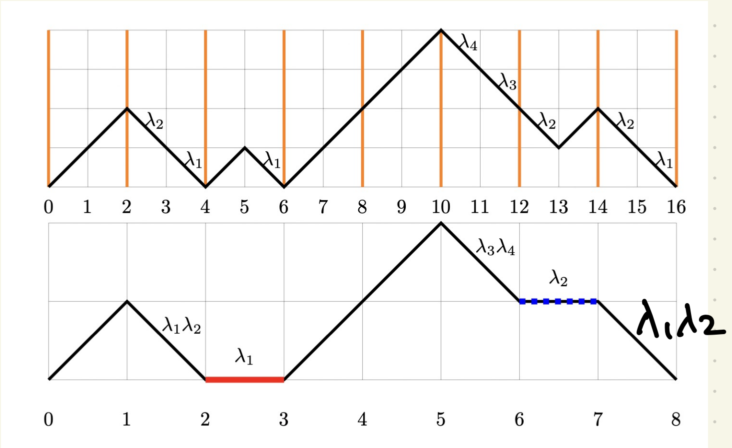
$\mathrm{Motz}_n^0(2)$ = set of all $\pi \in \mathrm{Motz}_n(2)$ such that $\pi$ has no $H_s$ on $x$-axis.

Let $\pi = S_1 \cdots S_{2n} \in \mathrm{Dyck}_{2n}$.

Define $\phi_0(\pi) = \tau \in \mathrm{Motz}_n^0(2)$,

$\tau = T_1 \cdots T_n$,

$$T_i = \begin{cases} U & \text{if } S_{2i-1}S_{2i} = UU \\ D & \text{"} \qquad DD \\ H_r & \text{"} \qquad UD \\ H_b & \text{"} \qquad DU \end{cases}$$

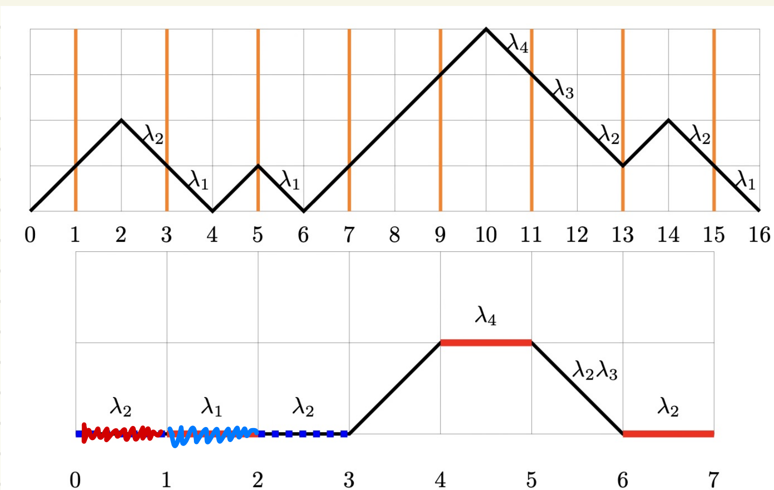


Let $\pi = s_1 s_2 \cdots s_{2n+2} \in \mathrm{Dyck}_{2n+2}$.

Define $\phi_1(\pi) = \tau \in \mathrm{Motz}_n(2)$

$\tau = T_1 \cdots T_n$,

$$T_i = \begin{cases} U & \text{if } s_{2i}s_{2i+1} = UU \\ D & \text{if } \quad '' \quad = DD \\ H_r & \qquad '' \quad = UD \\ H_u & \qquad '' \quad = DU \end{cases}$$




<u>Thm</u> $\phi_0 : \mathrm{Dyck}_{2n} \to \mathrm{Motz}_n^0(2)$

$\phi_1 : \mathrm{Dyck}_{2n+2} \to \mathrm{Motz}_n(2)$.

are bijections such that

$\phi_0(\pi) = \tau$

$\Rightarrow wt(\pi; \mathbf{0}, \boldsymbol{\lambda}) = wt(\tau; \mathbf{b}^e, \boldsymbol{\lambda}^e)$

$\phi_1(\pi) = \tau$

$\Rightarrow wt(\pi; \mathbf{0}, \boldsymbol{\lambda}) = \lambda_1 \, wt(\tau; \mathbf{b}^o, \boldsymbol{\lambda}^o)$.

<u>Cor</u>

$$\mu_{2n}(\mathbf{0}, \boldsymbol{\lambda}) = \mu_n(\mathbf{b}^e, \boldsymbol{\lambda}^e)$$

$$\mu_{2n+2}(\mathbf{0}, \boldsymbol{\lambda}) = \lambda_1 \mu_n(\mathbf{b}^o, \boldsymbol{\lambda}^o).$$

## §9.3. J-fractions and S-fractions.

An <u>S-fraction</u> (Stieltjes-fraction) is a continued fraction of form

$$\cfrac{1}{1-\cfrac{c_1 x}{1-\cfrac{c_2 x}{\ddots}}} = \sum_{n\geq 0} \sum_{\pi \in \mathrm{Dyck}_n} \mathrm{wt}(\pi; \mathbb{0}, \overset{\{c_1, c_2, \dots\}}{\overset{\shortparallel}{\mathbb{c}}})\, x^n$$

A <u>J-fraction</u> (Jacobi-fraction) is

$$\cfrac{1}{1-b_0 x-\cfrac{\lambda_1 x^2}{1-b_1 x-\cfrac{\lambda_2 x^2}{\ddots}}} = \sum_{n\geq 0} \sum_{\pi \in \mathrm{Motz}_n} \mathrm{wt}(\pi; \mathbb{b}, \boldsymbol{\lambda})\, x^n$$

<u>Thm</u>

$$\underset{①}{\frac{1}{1-\frac{\lambda_1 x}{1-\frac{\lambda_2 x}{\ddots}}}} = \frac{1}{1-(\lambda_0+\lambda_1)x-\frac{\lambda_1\lambda_2 x^2}{1-(\lambda_2+\lambda_3)x-\frac{\lambda_3\lambda_4 x^2}{\ddots}}} \quad ②$$

$$= 1+\frac{\lambda_1 x}{1-(\lambda_1+\lambda_2)x-\frac{\lambda_2\lambda_3 x^2}{1-(\lambda_3+\lambda_4)x-\frac{\lambda_4\lambda_5 x^2}{\ddots}}} \quad ③$$

$(\lambda_0=0)$

Pf) $① = \sum_{n\geq 0}\sum_{\pi\in \mathrm{Dyck}_{2n}} wt(\pi;0,\lambda)x^n = \sum_{n\geq 0}\mu_{2n}(0,\lambda)x^n$

$② = \sum_{n\geq 0}\sum_{\pi\in \mathrm{Motz}_n} wt(\pi;b^e,\lambda^e)x^n = \sum_{n\geq 0}\mu_n(b^e,\lambda^e)x^n$

$\lambda_1\sum_{n\geq 0}\mu_n(b^o,\lambda^o)x^n = \sum_{n\geq 0}\mu_{2n+2}(0,\lambda)x^n = \frac{1}{x}\left(\sum_{n\geq 0}\mu_{2n}(0,\lambda)x^n - 1\right)$

$\Rightarrow \underset{①}{\underset{\|}{\sum_{n\geq 0}\mu_{2n}(0,\lambda)x^n}} = 1+\lambda_1 x\sum_{n\geq 0}\mu_n(b^o,\lambda^o)x^n = 1+\lambda_1 x\sum_{n\geq 0}\sum_{\pi\in \mathrm{Motz}_n} wt(\pi;b^o,\lambda^o)x^n$

$= ③.$