Ch 10 Linearization coefficients.

$\{P_n(x)\}_{n\geq 0}$: a monic OPS
with linear functional $\mathcal{L}$.

$$P_m(x)P_n(x) = \sum_{\ell\geq 0} c_{m,n}^{(\ell)} P_\ell(x)$$

Multiply $P_\ell(x)$ and take $\mathcal{L}$

$$\mathcal{L}(P_\ell P_m P_n) = c_{m,n}^{(\ell)} \mathcal{L}(P_\ell^2).$$

$$\Rightarrow \quad c_{m,n}^{(\ell)} = \frac{\mathcal{L}(P_\ell P_m P_n)}{\lambda_1 \cdots \lambda_\ell}$$

Def) A linearization coeff is

$$\mathcal{L}(P_{n_1} P_{n_2} \cdots P_{n_k}).$$

Goal: Find combinatorial models for Lin. coeff

for Hermite
Charlier
Laguerre.

Idea: ① Find comb model for $P_n(x)$
② Find comb model for $\mu_n = \mathcal{L}(x^n)$.
③ $\mathcal{L}(P_{n_1} \cdots P_{n_k})$ = comb inter...

§10.1. Hermite poly

rescaled Hermite $\widetilde{H}_n(x)$

$$\widetilde{H}_{n+1}(x) = x\,\widetilde{H}_n(x) - n\,\widetilde{H}_{n-1}(x)$$

$(H_0 = 1,\ H_{-1} = 0)$.

$\mu_n = \mathcal{L}(x^n) = \#$ perfect matchings on $[n]$.

Def) M_n = set of matchings on $[n]$.

For $\tau \in M_n$, $e(\tau) = \#$ edges

$\mathrm{fix}(\tau) = \#$ fixed pts.

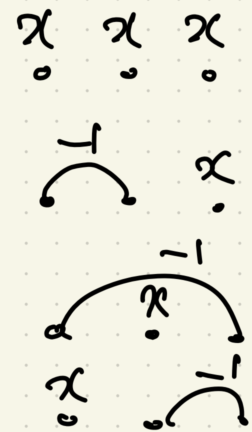
$e(\tau) = 2$, $\mathrm{fix}(\tau) = 3$.

Lem $\displaystyle \widetilde{H}_n(x) = \sum_{\tau \in M_n} (-1)^{e(\tau)} x^{\mathrm{fix}(\tau)}$.

Pf) Easy by ind. □

$n+1$

ex) $\widetilde{H}_3(x) = x^3 - 3x$

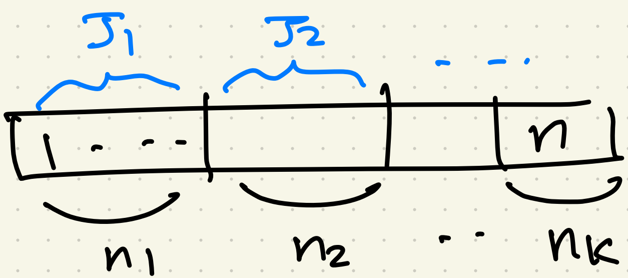


Def) $n_1, \dots, n_k$: fixed int. (≥ 0).

Let $n_1 + \cdots + n_k = n$.

For $1 \leq s \leq k$,

$$J_s := \{ i : n_1 + \cdots + n_{j-1} + 1 \leq i \leq n_1 + \cdots + n_j \}$$



$[n] = J_1 \sqcup \cdots \sqcup J_k$

Let $\tau \in M_n$.

An edge (i,j) of τ is <u>homogeneous</u> if $i, j \in J_s$.

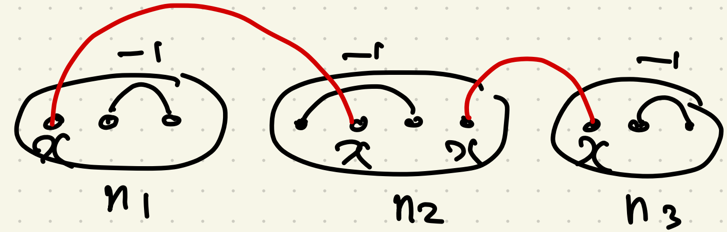
otherwise <u>inhomogeneous</u>

τ is <u>homogeneous</u> (inhomogeneous) if every edge is <u>homogeneous</u> (inhomogeneous).

$IPM(n_1, \dots, n_k)$
= set of all inhomogeneous perfect matchings on $[n]$.

Thm

$$\mathcal{L}(\widehat{H}_{n_1}\cdots\widehat{H}_{n_k}) = |IPM(n_1,\dots,n_k)|$$



Pf) $\mathcal{L}(\widehat{H}_{n_1}\cdots\widehat{H}_{n_k})$

$$= \mathcal{L}\left[\sum_{(\tau_1,\dots,\tau_k)\in M_{n_1}\times\cdots\times M_{n_k}} \prod_{i=1}^{k} (-1)^{e(\tau_i)} x^{\mathrm{fix}(\tau_i)}\right]$$

$$= \sum_{(\tau_1,\dots,\tau_k)} (-1)^{e(\tau_1)+\cdots+e(\tau_k)} \underbrace{\mathcal{L}\left(x^{\mathrm{fix}(\tau_1)+\cdots+\mathrm{fix}(\tau_k)}\right)}_{\substack{\shortparallel \\ \#\text{ perfect matchings} \\ \text{of size } \mathrm{fix}(\tau_1)+\cdots+\mathrm{fix}(\tau_k).}}$$

$\tau = \bigcup \tau_i$ (black edges)

π = perfect matching on fixed pts. (red edges)

τ : homogeneous.

$$= \sum_{(\tau,\pi)\in X} (-1)^{e(\tau)}$$

Let's find sign-rev. inv.

$\phi : X \to X$.

Let $(\tau, \pi) \in X$.

① There is no homogeneous edges in τ, π.

$$\phi(\tau, \pi) = (\tau, \pi).$$

fixed pts only IPM.

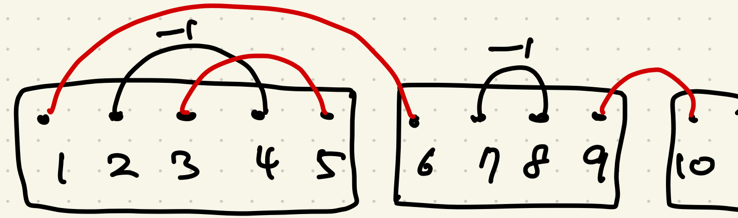
② There is a homo edge.

Let (i,j) be the homogeneous edge with smallest j.

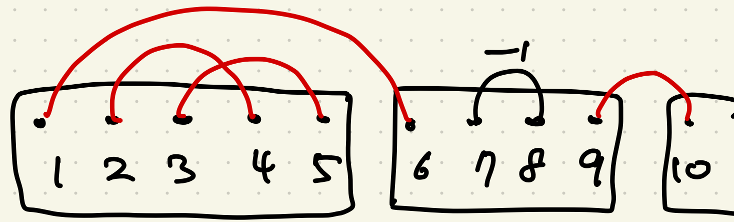
Change the color of (i,j)

Claim: $\phi : X \to X$ s.r.i.

Pf: Easy!

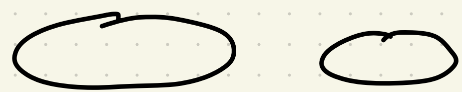


$n_1 = 5$ $n_2 = 4$ $n_3 = 1$.



$n_1 = 5$ $n_2 = 4$ $n_3 = 1$.

Cor $\mathcal{L}(\widetilde{H}_m \widetilde{H}_n) = \delta_{m,n}\, n!$



m n

§10.2. Charlier poly

$$C_{n+1}(x;a) = (x-n-a)\,C_n(x;a) - an\,C_{n-1}(x;a).$$

let's consider $C_n(x) = C_n(x;1)$.
($a=1$ case only).

$$C_{n+1}(x) = (x-n-1)\,C_n - n\,C_{n-1}(x).$$

$$b_n = n+1, \quad \lambda_n = n.$$

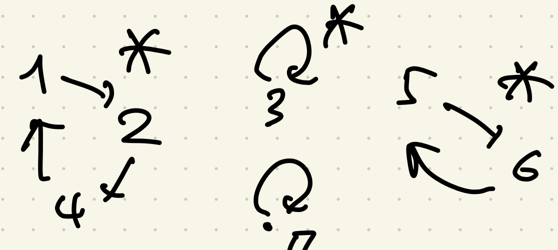
$$\mu_n = \mathcal{L}(x^n) = |\Pi_n|$$

Π_n: set of all set partitions of $[n]$.

Def A *decorated permutation* is a permutation in which every cycle may be decorated.

A *properly decorated* perm is a decorated perm such that every cycle of len ≥ 2 is decorated.

ex)



$D_n = \{\text{decor. perm on } [n]\}$

$PD_n = \{\text{prop. dec. } \quad " \quad \}$.

$dc(\pi) = \#$ decorated cycles.

Lem $C_n(x) = (-1)^n \sum_{\pi \in PD_n} (-x)^{dc(\pi)}$.

Pf) Let $F_n(x) = \text{RHS}$.

We will show $C_n = F_n$ by ind.

$n=0,1$: true.

Suppose it's true for $k \le n$.

Sufficient to show

$$F_{n+1} = \underbrace{(x-n-1)F_n}_{} - nF_{n-1}.$$

$$x F_n - F_n - nF_n - nF_{n-1}$$

$$= (-1)^n \left[-\sum_{\pi \in A_1} (-x)^{dc(\pi)} - \sum_{\pi \in A_2} (-x)^{dc(\pi)} - \sum_{A_3} + \sum_{A_4} \right]$$

$A_1 = \{ \pi \in D_{n+1} \mid \pi = \sigma\,(n+1)^*, \sigma \in PD_n \}$

$A_2 = \{ \pi \mid \pi = \sigma\,(n+1),$ " $\}$

$A_3 = \{ \pi \mid \pi$ is $\sigma \in PD_n$ with $n+1$ inserted after some int. $\}$

$A_4 = \{ \pi \mid \pi = \sigma'(i, n+1)$, $\sigma \in PD_{n-1}$, $\sigma' = \sigma$ with every $j \geq i$ increased by $1 \}$

$\pi \in A_3$

$n+1$

$\sigma = (\qquad)(\qquad)(\quad)$

$\in PD_n$.

π is not properly decor.

iff $\pi = \cdots \quad (i, n+1), \ \cdots$

$\Updownarrow$

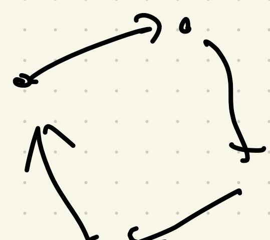
$A4$.

We are left with exactly PD_{n+1}.

Def) $n_1, \ldots, n_k$ fixed

A cycle is <u>homogeneous</u> if all elts are in J_s.

A cycle is <u>inhomogeneous</u> if every arrow connects two elts from J_r, J_s $(r \neq s)$



(every arc is inhom.).

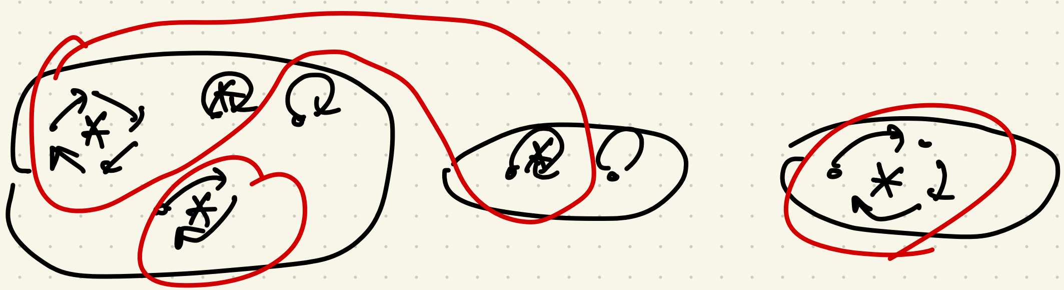
Def) $\Pi(n_1,\dots,n_k)$ = set of inhomogeneous set partitions.
(no singleto, every edge is inhomogeneow)

<u>Thm.</u> $(n=n_1+\cdots+n_k)$

$$\mathcal{L}(C_{n_1}\cdots C_{n_k}) = |\Pi(n_1,\dots,n_k)|$$

Pf) $$\text{LHS} = (-1)^n \mathcal{L}\left(\sum_{(\tau_1,\dots,\tau_k)\in\Pi_{n_1}\times\cdots\times\Pi_{n_k}} \prod_{i=1}^{k} (-x)^{dc(\tau_i)}\right)$$

$$= (-1)^n \sum_{(\tau_1,\dots,\tau_k)} (-1)^{dc(\tau_1)+\cdots+dc(\tau_k)} \mathcal{L}\left(x^{dc(\tau_1)+\cdots+dc(\tau_k)}\right)$$



let X = set of collections $\pi = \{B_1, \dots B_r\}$ s.t.

① $B_i = \{C_1^{(i)}, \dots, C_{t_i}^{(i)}\} \neq \phi$ homogeneous cycles.

② $\{C_j^{(i)} : 1 \le i \le r, \ 1 \le j \le t_i\}$ is a partition of a subset $A \subseteq [n]$.
↳ as a set

$\mathrm{sgn}(\pi) = (-1)^{|B_1| + \cdots + |B_r|}$.

$\Rightarrow \ \mathcal{L}(C_{n_1} \cdots C_{n_k}) = (-1)^n \sum_{\pi \in X} \mathrm{sgn}(\pi)$.

let's find s.r.i. $\phi : X \to X$. Suppose $\pi = \{B_1, \dots B_r\} \in X$.

① If $\exists$ non-dec cycle (i) or $\exists B_j = \{(i)^*\}$

$\Rightarrow$ change $(i) \longleftrightarrow \{(i)^*\}$ (i : smallest).

ex). $\tau = \{(1,3)^*, (2)^*, (4)\} \cup \{(5,6,9)^*, (7), (8)\} \cup \{(10)^*, (11,12)^*\}$

$$n_1 = 4 \qquad n_2 = 5 \qquad n_3 = 3$$

$$\pi = \boxed{(1,3)^*, (5,6,9)^*} \quad \boxed{(2)^*, (11,12)^*} \quad \boxed{(10)^*}$$

$$\Updownarrow$$

$$\tau = \{(1,3)^*, (2)^* \} \cup \{(5,6,9)^*, (7), (8)\} \cup \{(10)^*, (11,12)^*\}$$

$$n_1 = 4 \qquad n_2 = 5 \qquad n_3 = 3$$

$$\pi = \boxed{(1,3)^*, (5,6,9)^*} \quad \boxed{(2)^*, (11,12)^*} \quad \boxed{(10)^*}$$

$$\boxed{(4)^*}$$

② No (i), No $\boxed{(i)^*}$

Find lex smallest (i,j) s.t. i & j in same block.

B_s and J_r.

ex). $\tau = \{(1,3,4)^*, (2)^*\} \cup \{(5,6,9)^*, (7)^*, (8)^*\} \cup \{(10)^*, (11,12)^*\}$

$\pi = \boxed{(1,3,4)^*, (2)^*, (5,6,9)^*}\ \boxed{(7)^*, (8)^*, (10)^*}\ \boxed{(11,12)^*}$

$(i,j) = (1,2) \Rightarrow$ multiply $(1,2)$ to $(1,3,4)^*(2)^*$

$(1,2)(1,3,4)(2) = (1,3,4,2)$

$\tau = \{(1,3,4)^*, (2)^*\} \cup \{(5,6,9)^*, (7)^*, (8)^*\} \cup \{(10)^*, (11,12)^*\}$

$\pi = \boxed{(1,3,4,2)^*, (5,6,9)^*}\ \boxed{(7)^*, (8)^*, (10)^*}\ \boxed{(11,12)^*}$

Fix pts $\pi = \{B_1, \ldots, B_k\}$

① B_i has (dec) cycles of len 1

② $|B_i| \geq 2$

③ $B_i \cap J_j$ has at most one elt for all j.