§10.3. Tchebyshev polynomials.

skipped

§10.4. Laguerre polynomials

$$L_{n+1}^{(\alpha)}(x) = (x-2n-\alpha) L_n^{(\alpha)}(x) - n(n-1+\alpha) L_{n-1}^{(\alpha)}(x).$$

only consider $\alpha = 1$.

let $L_n(x) = L_n^{(1)}(x)$.

$$L_{n+1}(x) = (x-2n-1) L_n(x) - n^2 L_{n-1}(x).$$

$$\mu_n = \mathcal{L}(x^n) = n!$$

Def). $[n] = \{1, \dots, n\}$.

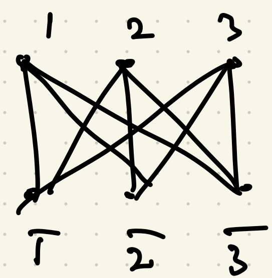
$[\overline{n}] = \{\overline{1}, \dots, \overline{n}\}$.

$K_{n,n}$ = the complete bipartite graph (V, E),

$V = [n] \sqcup [\overline{n}]$.

$E = \{(i, \overline{j}) : i, j \in [n]\}$.

e.g. $K_{3,3}$



$M(K_{n,n})$ = set of matchings on $K_{n,n}$.

Lem $L_n(x) = \sum_{\tau \in M(K_{n,n})} (-1)^{e(\tau)} x^{\mathrm{fix}(e)/2}$

pf) let $F_n(x) = $ RHS.

We will prove $L_n = F_n$ by ind.

$n=0,1$: true.

Suppose true for $k \le n$.

Enough to show

① $F_{n+1} = (x-2n-1)F_n - n^2 F_{n-1}$

let $w(\tau) = (-1)^{e(\tau)} x^{\mathrm{fix}(e)/2}$.

RHS of ① is

$xF_n - F_n - nF_n - nF_n - n^2 F_{n-1}$

$= \sum_{\tau \in A_1} w(\tau) + \cdots + \sum_{\tau \in A_4} w(\tau) - \sum_{\tau \in A_5} w(\tau)$

$A_1 = \{\tau \in M(K_{n+1,n+1}) : n+1, \overline{n+1} : \text{fixed pts}\}$

$A_2 = \{\tau : (n+1, \overline{n+1}) \text{ edge}\}$.

$A_3 = \{\tau : (\bar{i}, n+1) \text{ edge } 1 \le i \le n\}$.

$A_4 = \{\tau : (i, \overline{n+1})$ " " $\}$

$A_5 = \{\tau : (\bar{i}, n+1), (j, \overline{n+1}) \text{ edges } 1 \le i, j \le n\}$.

$A_3 \cap A_4 = A_5$

$① = \sum_{\tau \in A_1 \cup A_2 \cup A_3 \cup A_4} w(\tau)$.

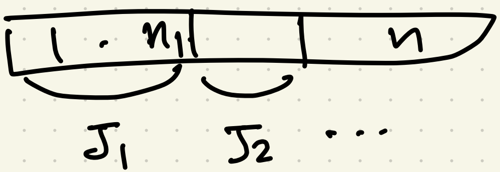
$= F_{n+1}$ $\square$

Recall A derangement is a perm $\pi \in S_n$ s.t. $\pi(i) \neq i$ $\forall i$.

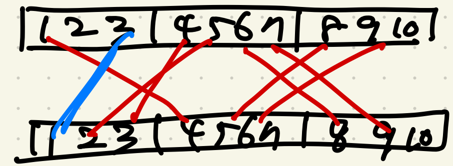
Def) $n_1, \dots, n_k \geq 0$. Fixed.

$n = n_1 + \dots + n_k$.

$J_1, \dots J_k$.



An $\underline{(n_1, \dots, n_k)\text{-derangement}}$ is a permutation $\pi \in S_n$ such that $\forall\, 1 \leq s \leq k$, if $i \in J_s$ then $\pi(i) \notin J_s$

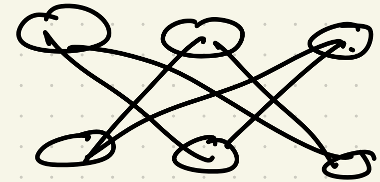


In other words, $(n_1, \dots, n_k)$-der. is an inhomogeneous perfect matching on $K_{n,n}$.

A usual der. is $(1, \dots 1)$-der.

$d(n_1, \dots n_k) = \#$ all $(n_1, \dots n_k)$-der.

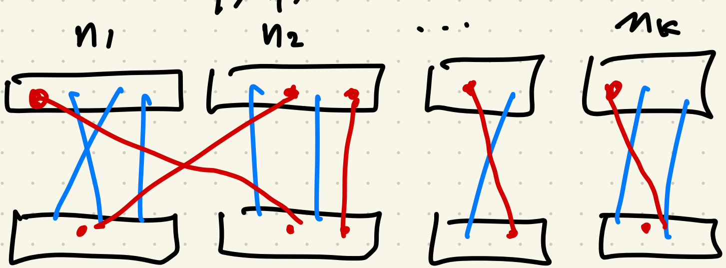
Thm $\mathcal{L}(L_{n_1}(x)\cdots L_{n_k}(x))$
$= d(n_1,\dots,n_k)$

fixed pts: 

Pf) $n = n_1+\cdots+n_k$.

$$\mathcal{L}(L_{n_1}\cdots L_{n_k})$$
$$= \mathcal{L}\left(\sum_{(\tau_1,\dots,\tau_k)\in M(K_{n_1},n_1)\times\cdots\times M(K_{n_k},n_k)} (-1)^{e(\tau_1)+\cdots+e(\tau_k)}\, x^{(\mathrm{fix}(\tau_1)+\cdots+\mathrm{fix}(\tau_k))/2}\right)$$

$$= \sum_{(\tau_1,\dots,\tau_k)\in M(K_{n_1},n_1)\times\cdots\times M(K_{n_k},n_k)} (-1)^{e(\tau_1)+\cdots+e(\tau_k)} \left((\mathrm{fix}(\tau_1)+\cdots+\mathrm{fix}(\tau_k))/2\right)!$$



blue edge: always homo.
-1

red: $+1$

□

§10.5. Multi-derangements and MacMahan's master theorem.

Goal: Prove this.

<u>Thm</u> $k>0$: fixed.

$$\sum_{n_1,\dots,n_k\geq 0} d(n_1,\dots,n_k)\frac{x_1^{n_1}}{n_1!}\cdots\frac{x_k^{n_k}}{n_k!}$$

$$=\frac{1}{1-e_2-2e_3-\cdots-(k-1)e_k}.$$

e_n = elementary sym ftn on $x_1,\dots,x_k$

$$=\sum_{1\leq i_1<\cdots<i_n\leq k} x_{i_1}\cdots x_{i_n}$$

MacMahon's master thm

$A=(a_{i,j})_{i,j=1}^{k}$ matrix.

$F(n_1,\dots n_k)$

$$=[y_1^{n_1}\cdots y_k^{n_k}]\prod_{i=1}^{k}(a_{i1}y_1+\cdots+a_{ik}y_k)^{n_k}$$

Then

$$\sum_{n_1,\dots,n_k\geq 0} F(n_1,\dots,n_k)x_1^{n_1}\cdots x_k^{n_k}$$

$$=\frac{1}{\det(I_k-TA)}$$

$$T=(\delta_{ij}\,x_i)_{i,j=1}^{k}$$

Def) A <u>multi-derangent</u> of

$\{1^{n_1}, 2^{n_2}, \ldots, k^{n_k}\}$ is

an arrangent

$$\pi = \begin{pmatrix} \overbrace{1 \cdots 1}^{n_1} & \overbrace{2 \cdots 2}^{n_2} & \cdots & \overbrace{k \cdots k}^{n_k} \\ \pi_1 \cdots & & & \pi_n \end{pmatrix}$$

of the elements

$$\underbrace{1 \cdots 1}_{n_1} \; \underbrace{2 \cdots 2}_{n_2} \; \cdots \; \underbrace{k \cdots k}_{n_k}$$

such that

$$\pi(i) \neq i \quad \forall\, 1 \leq i \leq k.$$

Let $m(n_1, \ldots, n_k)$ be

multi-der of $\{1^{n_1}, \ldots, k^{n_k}\}$.

Then

$$m(n_1, \ldots n_k) = \frac{d(n_1, \ldots n_k)}{n_1! \cdots n_k!}$$

Lem. $a_{ij} = 1 - \delta_{ij}$. $A = (a_{ij}) = \underset{\sim}{J} - I$
all 1 mat.

$m(n_1, \dots, n_k)$

$= [y_1^{n_1} \cdots y_k^{n_k}] \prod_{i=1}^{n} (a_{i1} y_1 + \cdots + a_{ik} y_k)^{n_i}$

Pf) $\pi = \pi_1 \cdots \pi_n$: multi-der of $\{1^{n_1}, \dots, k^{n_k}\}$.

$$\pi = \begin{pmatrix} 1 & \cdots & 1 & 2 & \cdots & 2 & \cdots \\ \boxed{\pi_1 & \cdots & \pi_{n_1}} & \underbrace{\pi_{n_1+1} & \cdots & \pi_{n_1+n_2}} & & & \cdots \end{pmatrix}$$

↳ anything but 1

constructing this part is the same as expanding $(a_{11} y_1 + \cdots + a_{1k} y_k)^{n_1}$
$= (0 \cdot y_1 + y_2 + \cdots + y_k)^{n_1}$

Second part: $\pi_{n_1+1} \cdots \pi_{n_1+n_2}$ can be constructed by expanding

$(y_1 + 0 \cdot y_2 + y_3 + \cdots + y_k)^{n_2}$

$\vdots$

If we collect the terms $y_1^{n_1} \cdots y_k^{n_k}$, we get all multi-der of $\{1^{n_1}, \dots, k^{n_k}\}$.

□

let J_k = $k \times k$ all-1 matrix.

Lem $\det(I_k - J_k) = 1 - k$.

Pf). let $\lambda_1, \dots, \lambda_k$ be the eig vals of J_k.

$\Rightarrow \det(xI_k - J_k) = (x-\lambda_1)\cdots(x-\lambda_k)$

J_k has eig val 0 with $k-1$ eig vec. $\varepsilon_i - \varepsilon_{i+1}$

k is an eig val with eig vec $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$(\lambda_1, \dots, \lambda_k) = (k, 0, \dots, 0)$

$\Rightarrow \det(xI_k - J_k) = x^{k-1}(x-k)$.

Lem $A = J_k - J_k$. $T = \mathrm{diag}(x_1, \dots, x_k)$

$\Rightarrow \det(I_k - TA) = 1 - e_2 - 2e_3 - \cdots - (k-1)e_k$

Pf) $I_k - TA = (b_{ij})_{i,j=1}^{k}$.

$$b_{ij} = \begin{cases} 1 & \text{if } i = j \\ -x_i & \text{otherwise.} \end{cases}$$

$$\det(I_k - TA) = \sum_{\pi \in S_k} \mathrm{sgn}(\pi) \prod_{i=1}^{k} b_{i,\pi(i)}.$$

$$= \sum_{H \subseteq [k]} \sum_{\substack{\pi \in S_k \\ \mathrm{Fix}(\pi) = [k] - H.}} \mathrm{sgn}(\pi) \prod_{i \in H} b_{i,\pi(i)}$$

$$= \sum_{H \subseteq [k]} \det(I_{|H|} - J_{|H|}) \prod_{i \in H} x_i$$

$$= \sum_{i=0}^{k} (1-i) e_i \qquad \square.$$

Pf of Thm

Let $A = J_k - I_k$.

Then

MMT & Lem

$$\sum_{n_1,\dots,n_k \geq 0} m(n_1,\dots,n_k)\, x_1^{n_1} \cdots x_k^{n_k} = \frac{1}{\det(I - TA)}$$

$$= \frac{1}{1 - e_2 - 2e_3 - \cdots - (k-1)e_k} \quad \square.$$

Lem.