§2.2. The moment functional and orthogonality

$\mathbb{C}[x]$ = the space of polynomials in x with coeffs in $\mathbb{C}$.

A <u>linear functional</u> on $\mathbb{C}[x]$ is a map $\mathcal{L}: \mathbb{C}[x] \to \mathbb{C}$ such that

$$\mathcal{L}(af(x)+bg(x)) = a\mathcal{L}(f(x)) + b\mathcal{L}(g(x))$$

for all $f(x), g(x) \in \mathbb{C}[x]$, $a, b \in \mathbb{C}$.

Def) $\{\mu_n\}_{n\geq 0}$: seq of complex num.
$\mathcal{L}$: the lin. functional on $\mathbb{C}[x]$. defined by $\mathcal{L}(x^n) = \mu_n$.

We say that $\mathcal{L}$ is the <u>moment functional</u> determined by moment seq $\{\mu_n\}$.
μ_n is called the <u>nth moment</u> of $\mathcal{L}$.

Def) $\mathcal{L}$: a lin functional on $\mathbb{C}[x]$.
$\{P_n(x)\}_{n\geq 0}$ is an orthogonal polynomial sequence (OPS) w.r.t. $\mathcal{L}$ if

① $\deg P_n(x) = n \quad \forall n \geq 0$

② $\mathcal{L}(P_m(x)P_n(x)) = K_n\delta_{m,n}$, ($K_n \neq 0$) for some.

We say $\{P_n(x)\}$ is <u>orthonormal</u> if

$$\mathcal{L}(P_m(x)P_n(x)) = \delta_{m,n}$$

From now on, we will always assume

$$\deg P_n(x) = n.$$

Thm $\{P_n(x)\}$ is a seq of poly.

$\mathcal{L}$: lin functional.

TFAE.

① $\{P_n(x)\}$ OPS for $\mathcal{L}$.

② $\mathcal{L}(\pi(x)P_n(x)) = 0$ if $\deg \pi(x) < n$
$\neq 0$ if $\deg \pi(x) = n$.

③ $\mathcal{L}(x^m P_n(x)) = K_n \delta_{m,n}$, $0 \le m \le n$
for some $K_n \neq 0$.

Pf) ① ⇒ ② : Suppose $\deg \pi(x) \le n$.

$$\pi(x) = \sum_{k=0}^{n} a_k P_k(x).$$

$$\mathcal{L}(\pi(x)P_n(x)) = \mathcal{L}\left(\sum_{k=0}^{n} a_k P_k(x) P_n(x)\right)$$

$$= \sum_{k=0}^{n} a_k \mathcal{L}(P_k(x)P_n(x)).$$

$$= a_n K_n \qquad (K_n \neq 0).$$

↳ zero if $\deg \pi(x) < n$
nonzero if " $= n$.

② ⇒ ③ : Just take $\pi(x) = x^m$.

③ ⇒ ① : Easy! □

Thm Suppose $\{P_n(x)\}$: OPS for $\mathcal{L}$.
and $\pi(x)$: poly of deg n.

$$\pi(x) = \sum_{k=0}^{n} a_k P_k(x), \quad a_k = \frac{\mathcal{L}(\pi(x)P_k(x))}{\mathcal{L}(P_k(x)^2)}.$$

Pf) Multiply both sides by $P_j(x)$
and take $\mathcal{L}$.

$$\mathcal{L}(\pi(x)P_j(x)) = \sum_{k=0}^{n} a_k \mathcal{L}(P_k(x)P_j(x)).$$

$$= a_j \mathcal{L}(P_j(x)^2)$$

$$\Rightarrow a_j = \frac{\mathcal{L}(\pi(x)P_j(x))}{\mathcal{L}(P_j(x)^2)}.$$

□

Thm. $\{p_n(x)\}$: OPS for $\mathcal{L}$.

$\Rightarrow$ $p_n(x)$ is uniquely determined by $\mathcal{L}$ up to a nonzero scalar mult.

More precisely, if $\{Q_n(x)\}$ is OPS for $\mathcal{L}$, then $Q_n(x) = c_n p_n(x)$ for some $c_n \neq 0$.

pf) Let $Q_n(x) = \sum_{k=0}^{n} c_k p_k(x)$.

$$\Rightarrow c_k = \frac{\mathcal{L}(p_k(x) Q_n(x))}{\mathcal{L}(p_k(x)^2)}$$

$\hookrightarrow$ zero if $k<n$ and nonzero if $k=n$.

$$\Rightarrow Q_n(x) = c_n p_n(x). \qquad \square$$

Note: If $\{p_n(x)\}$ is OPS for $\mathcal{L}$ then it is also OPS for $\mathcal{L}' = c\mathcal{L}$ $(c \neq 0)$

So we may assume $\mathcal{L}(1) = 1$.

Note If $\{p_n(x)\}$ is OPS for $\mathcal{L}$ then $\{c_n p_n(x)\}$ " $(c_n \neq 0)$.

We can always find a monic OPS for $\mathcal{L}$.

leading coeff $=1$.

In fact, $\exists$ unique monic OPS for $\mathcal{L}$.

Also, $\exists$ orthonormal OPS for $\mathcal{L}$ by letting $\hat{p}_n(x) = \dfrac{p_n(x)}{\mathcal{L}(p_n(x)^2)^{\frac{1}{2}}}$

Cor Suppose $\mathcal{L}$ is a lin. ftnl with some OPS. Let $\{K_n\}_{n\geq 0}$ be a seq of nonzero numbers.

① $\exists$ unique monic OPS for $\mathcal{L}$.

② $\exists$ " OPS $\{p_n(x)\}$ " s.t leading coeff of $p_n(x) = K_n$.

③ $\exists$ unique OPS $\{p_n(x)\}$ for $\mathcal{L}$ s.t $\mathcal{L}(x^n p_n(x)) = K_n$.

§2.3. Existence of OPS.

Q: For what $\mathcal{L}$ does there exist OPS?

Def) The <u>Hankel determinant</u> of a moment sequence $\{\mu_n\}_{n\geq 0}$ is

$$\Delta_n = \det(\mu_{i+j})_{i,j=0}^{n} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}$$

<u>Thm</u> $\mathcal{L}$: lin. ftnl with moment seq $\{\mu_n\}$. There is OPS for $\mathcal{L}$ iff $\Delta_n \neq 0$ $\forall n \geq 0$.

Pf) Fix a seq of nonzero numbers K_n, $n\geq 0$.

By Cor, if $\exists$ OPS for $\mathcal{L}$, there is a unique $\{P_n(x)\}$ OPS for $\mathcal{L}$ s.t. $\mathcal{L}(x^m P_n(x)) = K_n \delta_{m,n}$ for $0 \leq m \leq n$.

let $P_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$.

Mult x^m both sides and take $\mathcal{L}$.

$$\mathcal{L}(x^m P_n(x)) = \sum_{k=0}^{n} c_{n,k}\, \mu_{m+k} \overset{\text{hold}}{=} K_n \delta_{m,n}$$

We want to find $c_{n,k}$ s.t. hold.

$$\begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix} \begin{pmatrix} c_{n,0} \\ c_{n,1} \\ \vdots \\ c_{n,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ K_n \end{pmatrix}$$

$\exists$ unique sol in $c_{n,k}$ $\Longleftrightarrow$ $\Delta_n \neq 0$. $n \geq 0$.

We can solve the mat eq. using Cramer's rule

$$c_{n,n} = \frac{K_n \Delta_{n-1}}{\Delta_n} \neq 0.$$

$\Rightarrow$ $\deg P_n(x) = n$ (if $\Delta_n \neq 0$). $\square$

Lem $\{P_n(x)\}$: OPS for $\mathcal{L}$.
$\pi(x)$ has deg n.

$$\Rightarrow \mathcal{L}(\pi(x)P_n(x)) = \frac{ab\Delta_n}{\Delta_{n-1}},$$

$a =$ leading coeff of $\pi(x)$
$b =$ " $P_n(x)$.

In particular, if $\{P_n(x)\}$ is monic, $\mathcal{L}(P_n(x)^2) = \frac{\Delta_n}{\Delta_{n-1}}$.

Pf) We know from proof prev thm,

$$b = c_{n,n} = \frac{K_n \Delta_{n-1}}{\Delta_n}. \quad \left(K_n = \frac{b\Delta_n}{\Delta_{n-1}}\right)$$

let $\pi(x) = \sum_{k=0}^{n} a_k x^k \quad (a_n = a)$

$$\mathcal{L}(\pi(x)P_n(x)) = \sum_{k=0}^{n} a_k \mathcal{L}(x^k P_n(x))$$
$$= a_n \mathcal{L}(x^n P_n(x)) = a_n K_n = \frac{ab\Delta_n}{\Delta_{n-1}} \quad \square$$

Thm $\mathcal{L}$: lin ftnl with mom $\{\mu_n\}$.
Suppose $\Delta_n \neq 0 \ \forall n \geq 0$.
$\Rightarrow$ The monic OPS for $\mathcal{L}$ is

$$P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}$$

Pf) This can be done using Cramer's rule.
Alternatively, it's sufficient to show

$$\mathcal{L}(x^m P_n(x)) = \delta_{m,n} K_n \quad (K_n \neq 0,\ m \leq n)$$

$$\mathcal{L}\left(\frac{1}{\Delta_{n-1}} \begin{vmatrix} & & \\ & \ddots & \\ x^m & x^{m+1} & \cdots \end{vmatrix}\right) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} & & \\ & \ddots & \\ \mu_m & \mu_{m+1} & \cdots \end{vmatrix}$$

$$= \begin{cases} 0 & \text{if } m < n \text{ (two identical rows)} \\ \Delta_n/\Delta_{n-1} & \text{if } m = n. \end{cases} \quad \square$$

In many cases, there is a weight function $w(x)$ s.t.

$$\mathcal{L}(x^n) = \int_a^b x^n w(x)\,dx.$$

More generally, $\exists$ a measure ψ (ψ: non-decreasing)

$$\mathcal{L}(x^n) = \int_{-\infty}^{\infty} x^n \, d\psi(x)$$

<u>Fact</u>: Such an expression exists iff $\mathcal{L}(\pi(x)) > 0$ for every poly $\pi(x)$ s.t. $\pi(x) \geq 0$ $\forall x \in \mathbb{R}$ ($\pi(x) \not\equiv 0$). ($*$)

Def). A linear functional $\mathcal{L}$ is <u>positive-definite</u> if ($*$) holds.

<u>Thm</u> If $\mathcal{L}$ is pos-def, then $\exists$ real OPS for $\mathcal{L}$.

pf). First let's prove $\mu_n \in \mathbb{R}$.

Since $\mathcal{L}$ pos-def, $\mu_{2n} = \mathcal{L}(x^{2n}) > 0$.

$\mathcal{L}((x+1)^{2n}) > 0 \Rightarrow \mu_{2n-1} \in \mathbb{R}$. (by ind).

Let's construct real OPS $\{P_n(x)\}$.

Let $P_0(x) = 1$. $\in \mathbb{R}[x]$.

Suppose $P_0(x), \ldots, P_n(x)$ have been constructed

(This means $\mathcal{L}(P_i P_j) = 0$ unless $i \neq j$, $i,j \leq n$)

Let $P_{n+1}(x) = x^{n+1} + \sum_{k=0}^{n} a_k P_k(x)$. $\cdots$ ($*$)

We want: $\mathcal{L}(P_m P_{n+1}) = 0$ if $m \leq n$.

Mult P_m and take $\mathcal{L}$ in ($*$).

$$\mathcal{L}(P_m(x) P_{n+1}(x)) = \mathcal{L}(x^{n+1} P_m(x)) + a_m \mathcal{L}(P_m(x)^2).$$

This will be zero if $a_m = \dfrac{-\mathcal{L}(x^{n+1} P_m)}{\mathcal{L}(P_m^2)}$

By defining a_m in this way

we set $P_{n+1}(x) \in \mathbb{R}[x]$

and $\{P_0, \ldots, P_{n+1}\}$ real OPS.

We are done by ind. $\square$.