By defining a_m in this way we set $P_{n+1}(x) \in \mathbb{R}[x]$ and $\{P_0, \dots, P_{n+1}\}$ real OPS. We are done by ind. $\square$

Def) A polynomial $\pi(x)$ is <u>nonnegative-valued</u> if $\pi(x) \geq 0 \ \forall x \in \mathbb{R}$.

So, $\mathcal{L}$ is pos-def $\Leftrightarrow$ $\mathcal{L}(\pi(x)) > 0$ for all nonzero nonneg-val poly $\pi(x)$.

<u>Lem</u> Let $\pi(x)$ be a nonneg-val poly.

$\Rightarrow$ $\pi(x) = p(x)^2 + q(x)^2$ for some real poly $p(x), q(x)$.

Pf) Since $\pi(x) \in \mathbb{R}$ for every $x \in \mathbb{R}$, $\pi(x)$ is a real poly. ($\because$ lead coeff of $\pi(x)$ is $\lim_{x\to\infty} \frac{\pi(x)}{x^n}$).

Since $\pi(x) \geq 0$, $x \in \mathbb{R}$, all real roots of $\pi(x)$ have even multiplicity, and its complex roots appear in conjugate pairs.

$$\pi(x) = r(x)^2 \prod_{k=1}^{m} (x - \alpha_k - \beta_k i)(x - \alpha_k + \beta_k i).$$

$r(x) \in \mathbb{R}[x]$, $\alpha_k, \beta_k \in \mathbb{R}$.

Let $\prod_{k=1}^{m} (x - \alpha_k - \beta_k i) = A(x) + iB(x)$, $A(x), B(x) \in \mathbb{R}[x]$

Then $\prod_{k=1}^{m} (x - \alpha_k + \beta_k i) = A(x) - iB(x)$.

$$\Rightarrow \pi(x) = r(x)^2 (A(x) + iB(x))(A(x) - iB(x))$$
$$= r(x)^2 (A(x)^2 + B(x)^2) \quad \square$$

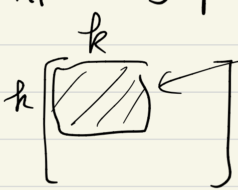
By lem, $\mathcal{L}$ is pos-def

$\iff \mathcal{L}(p(x)^2)>0$ for any nonzero poly $p(x)$.

Q: Why $\mathcal{L}$ is called pos-def?

Recall: A real $n\times n$ matrix A is pos-def if $u^TAu>0$ for any $u\in\mathbb{R}^n$, $u\neq 0$.

Sylvester's criterion says A is pos-def iff every principal minor of $A>0$. (and $A_{ij}\in\mathbb{R}$).



det of this is a prin. minor.

Thm $\mathcal{L}$ is pos-def $\iff$ $\mu_n\in\mathbb{R}$ and the Hankel matrix $\underbrace{(\mu_{i+j})_{i,j=0}^{n} \text{ is pos-def } \forall n\geq 0}_{\iff \Delta_n>0}$.

monic OPS

Pf) ($\Rightarrow$) By prev lem $\mathcal{L}(P_n(x)^2)=\dfrac{\Delta_n}{\Delta_{n-1}}$ (*)

Since $\mathcal{L}$ is pos-def, $\Delta_n/\Delta_{n-1}>0$

But $\mathcal{L}(P_0(x)^2)=\mathcal{L}(1)=\Delta_0>0$.

And $\Delta_n=\mathcal{L}(P_n(x)^2)\,\Delta_{n-1}>0$ for $n\geq 1$.

($\Leftarrow$) Since $\Delta_n\neq 0$, there is monic OPS $\{P_n(x)\}_{n\geq 0}$.

It's enough to show $\mathcal{L}(p(x)^2)>0$ for any nonzero poly $p(x)$.

Write $p(x)=\sum_{n=0}^{m} a_n P_n(x)$. $\left(\begin{array}{l} a_m\neq 0 \\ \deg p = m \end{array}\right)$.

$$\mathcal{L}(p(x)^2)=\mathcal{L}\left[\sum_{i=1}^{m} a_i P_i(x) \sum_{j=1}^{m} a_j P_j(x)\right]$$

$$=\sum_{i=1}^{m} a_i^2 \underbrace{\mathcal{L}(P_i(x)^2)}_{>0 \text{ by } (*)}$$

$$>0.$$

$\square$

§2.4. The fundamental recurrence.

<u>Thm</u> $\mathcal{L}$: lin ftnl with monic OPS $\{p_n(x)\}_{n\ge 0}$

$\Rightarrow$ $p_n(x)$ satisfy 3-term recurrence

$$(*) \quad p_{n+1}(x) = (x-b_n)\,p_n(x) - \lambda_n\, p_{n-1}(x)$$

for some seq $\{b_n\}_{n\ge 0}$, $\{\lambda_n\}_{n\ge 1}$.

with initial cond $p_{-1}(x)=0$, $p_0(x)=1$.

with $\lambda_n \ne 0$.

pf) Since $p_n(x)$ are monic,

$$p_{n+1}(x) - x\,p_n(x) = \sum_{i=0}^{n} a_i\, p_i(x).$$

It's enough to show $a_i = 0$ if $i \le n-2$.

Let $0 \le j \le n-2$. Mult $p_j(x)$ both sides and take $\mathcal{L}$,

$$\begin{aligned}\mathcal{L}(p_j\, p_{n+1} - (x p_j) p_n) &= \sum_{i=0}^{n} a_i\, \mathcal{L}(p_j p_i) \\ 0 \qquad &= a_j \underbrace{\mathcal{L}(p_j^2)}_{\ne 0}\end{aligned}$$

$\Rightarrow a_j = 0$.

It remains to show $\lambda_n \ne 0$.

Multiply x^{n-1} to $(*)$ and take $\mathcal{L}$.

$$\begin{aligned}\underset{0}{\cancel{\mathcal{L}(x^{n-1} p_{n+1})}} &= \mathcal{L}(x^n p_n) - b_n \overset{0}{\cancel{\mathcal{L}(x^{n-1} p_n)}} \\ &\quad - \lambda_n\, \mathcal{L}(x^{n-1} p_{n-1}).\end{aligned}$$

$$\begin{array}{ccccc} \Rightarrow & \mathcal{L}(x^n p_n) & = & \lambda_n\, \mathcal{L}(x^{n-1} p_{n-1}) & \\ & \| & & \| & \\ 0 \ne & \mathcal{L}(p_n p_n) & & \lambda_n\, \mathcal{L}(p_{n-1} p_{n-1}) & \ne 0 \end{array}$$

$\Rightarrow \lambda_n \ne 0$. $\square$

Thm $\mathcal{L}$: lin ftnl with monic OPS $\{P_n(x)\}_{n\ge 0}$

(*) $P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x)$

$\Rightarrow$ ① $\lambda_n = \dfrac{\mathcal{L}(P_n^2)}{\mathcal{L}(P_{n-1}^2)} = \dfrac{\Delta_{n-2}\Delta_n}{\Delta_{n-1}^2}$

② $b_n = \dfrac{\mathcal{L}(xP_n^2)}{\mathcal{L}(P_n^2)}$

③ $\mathcal{L}(P_n(x)^2) = \lambda_1 \cdots \lambda_n \mathcal{L}(1) = \dfrac{\Delta_n}{\Delta_{n-1}}$

④ $\Delta_n = \lambda_1^n \lambda_2^{n-1} \cdots \lambda_n^1 \mathcal{L}(1)^{n+1}$

Pf) We proved $\lambda_n = \dfrac{\mathcal{L}(P_n^2)}{\mathcal{L}(P_{n-1}^2)}$.

We also proved $\mathcal{L}(P_n^2) = \dfrac{\Delta_n}{\Delta_{n-1}}$.

$\Rightarrow$ ① holds.

Mult P_n and take $\mathcal{L}$ in (*)

$$\underbrace{\cancel{\mathcal{L}(P_nP_{n+1})}}_{0} = \mathcal{L}(xP_n^2) - b_n\mathcal{L}(P_n^2) - \lambda_n \underbrace{\cancel{\mathcal{L}(P_nP_{n-1})}}_{0}$$

$\Rightarrow$ ②.

③ follows from ①

④ " ③.

$\square$

Cor Suppose $\mathcal{L}$ has monic OPS. $\{P_n(x)\}$ $\boxed{\mathcal{L}(1)>0}$

$\mathcal{L}$ is pos-def $\iff$ $b_n \in \mathbb{R}$, $\lambda_n > 0$. $\forall n$.

Pf) ($\Rightarrow$) $P_n(x)$: real.

rec coeff $b_n, \lambda_n \in \mathbb{R}$.

By ①, $\lambda_n > 0$.

($\Leftarrow$) Since $b_n, \lambda_n \in \mathbb{R}$, $P_n(x)$ real.

It's easy to see $\mu_n \in \mathbb{R}$.

($\because$ $\mathcal{L}(P_n(x)) = 0$, $n \ge 1$).

By ④, $\Delta_n > 0$. $\square$

ex). Tchebyshev $T_n(x)$ is defined

by $T_n(\cos\theta) = \cos n\theta$.

$(n \geq 1)$

$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$

$T_{n+1} + T_{n-1} = 2x T_n$

$\circledast$ --- $T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$. $(n \geq 1)$

$T_1(x) = x T_0(x)$

$(T_0(x) = 1,\ T_1(x) = x)$.

lead coeff of $T_n(x)$ is $\begin{cases} 2^{n-1} & (n \geq 1) \\ 1 & n = 0. \end{cases}$

Define $\hat{T}_n(x) = \begin{cases} 2^{1-n} T_n(x) & (n \geq 1) \\ T_0(x) = 1 & (n = 0). \end{cases}$

$\hat{T}_n(x)$: monic Tchebychev.

Divide $\circledast$ by 2^n

$2^{-n} T_{n+1} = x \cdot 2^{1-n} T_n - 2^{-2} 2^{2-n} T_{n-1}$

$\hat{T}_{n+1} = x \hat{T}_n - \frac{1}{4} \hat{T}_{n-1} \quad (n \geq 2)$.

$\hat{T}_2 = x \hat{T}_1 - \frac{1}{2} \hat{T}_0 \quad (n=1)$

$T_0 = 1,\ T_1 = x,\ T_2 = 2x^2 - 1$

$\hat{T}_0 = 1,\ \hat{T}_1 = x,\ \hat{T}_2 = x^2 - \frac{1}{2}$.

$\hat{T}_{n+1}(x) = (x - b_n)\hat{T}_n(x) - \lambda_n \hat{T}_{n-1} \quad (n \geq 1)$.

$b_n = 0,\quad \lambda_n = \begin{cases} \frac{1}{4} & \text{if } n \geq 2. \\ \frac{1}{2} & \text{if } n = 1 \end{cases}$

If $b_n = 0$ then

$P_{2n}(x)$ is even function

$P_{2n+1}(x)$ is odd "

$$\begin{cases} P_{2n}(x) = P_{2n}(-x) \\ P_{2n+1}(-x) = -P_{2n+1}(x). \end{cases}$$

Def) $\mathcal{L}$ is <u>symmetric</u> if all of its odd moments are zero. ($\mu_{2n+1} = 0$).

<u>Thm</u> $\mathcal{L}$: lin ftnl with monic OPS $\{P_n(x)\}$. TFEA.

① $\mathcal{L}$ symmetric

② $P_n(-x) = (-1)^n P_n(x)$

③ $b_n = 0 \quad \forall n \geq 0$.

Pf) ① $\Rightarrow$ ② : $\mathcal{L}$ sym $\Rightarrow$ $\mathcal{L}(\pi(-x)) = \mathcal{L}(\pi(x))$ for all poly $\pi(x)$.

Thus $\mathcal{L}(P_m(-x)P_n(-x)) = \mathcal{L}(P_m(x)P_n(x)) = K_n \delta_{mn}$, $K_n \neq 0$.

$\Rightarrow \{P_n(-x)\}$ OPS for $\mathcal{L}$.

$\Rightarrow \quad P_n(-x) = c_n P_n(x) \Rightarrow c_n = (-1)^n$

② $\Rightarrow$ ① : Since $P_{2n+1}(-x) = -P_{2n+1}(x)$ $P_{2n+1}(x)$ is odd.

$\Rightarrow \underbrace{\mathcal{L}(P_{2n+1}(x))}_{=0} = $ sum of odd moments $= \mu_{2n+1} + $ (lower odd mom)

$\Rightarrow$ By ind, $\mu_{2n+1} = 0 \quad \forall n$.

② $\Leftrightarrow$ ③ : let $Q_n(x) = (-1)^n P_n(-x)$.

② means $P_n(x) = Q_n(x)$.

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x)$$

$$Q_n(x) = (-1)^n P_n(-x).$$

replace x by $-x$ multiply $(-1)^{n+1}$

$$(-1)^{n+1} P_{n+1}(-x) = (-x-b_n)(-1)^{n+1} P_n(-x) - \lambda_n (-1)^{n+1} P_{n-1}(-x)$$

$$Q_{n+1}(x) = (x+b_n)Q_n(x) - \lambda_n Q_{n-1}(x).$$

Thus $P_n(x) = Q_n(x) \iff b_n = 0.$ $\square$