Thm (Favard's thm).

$\{P_n(x)\}$: seq of monic poly.

$\exists \mathcal{L}$ s.t. $\{P_n(x)\}$ is OPS for $\mathcal{L}$

$\iff$

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x)$$

for some b_n, λ_n with $\lambda_n \neq 0$.

pf) $(\Rightarrow)$: We have already done.

$(\Leftarrow)$ If there is such $\mathcal{L}$,

we must have $\mathcal{L}(P_n(x)) = 0$, $n \geq 1$.

Therefore $\mathcal{L}$ is uniquely determined by

$(*)$ $\mathcal{L}(P_n(x)) = \delta_{n0}$, $n \geq 0$

So we define a lin ftnl $\mathcal{L}$ by $(*)$.

Enough to show for $0 \le k \le n$

Claim: $\mathcal{L}(x^k P_n(x)) = \delta_{k,n} \lambda_1 \cdots \lambda_n$.

We will prove this by induction on k.

If $k=0$, $\mathcal{L}(P_n(x)) = \delta_{n,0}$ true!

let $k \geq 1$. suppose claim is true for $k-1$.

Consider $n \geq k$.

Multiply x^{k-1} both sides of 3-term rec.

$$\Rightarrow x^k P_n(x) = x^{k-1} P_{n+1}(x) + b_n x^{k-1} P_n(x) + \lambda_n x^{k-1} P_{n-1}(x)$$

$$\mathcal{L}(x^k P_n(x)) = 0 + 0 + \lambda_n \mathcal{L}(x^{k-1} P_{n-1}(x))$$

$$= \lambda_n \cdot \delta_{n,k} \lambda_1 \cdots \lambda_{n-1}$$

$\Rightarrow$ True for k.

Claim is proved by induction. $\square$

Note $\mathcal{L}$ is pos-def iff $\lambda_n > 0$.

§2.5. Christoffel-Darboux Identities and zeros of orthogonal polynomials

<u>Thm</u> (Christoffel-Darboux Identities)

$\{p_n(x)\}$: poly satisfying

$$\circledast : \ p_{n+1}(x) = (x-b_n)p_n(x) - \lambda_n p_{n-1}(x). \quad (\lambda_n \neq 0)$$

① $$\sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{\lambda_1\cdots\lambda_k} = \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{\lambda_1\cdots\lambda_n(x-y)}$$

② $$\sum_{k=0}^{n} \frac{p_k(x)^2}{\lambda_1\cdots\lambda_k} = \frac{p_{n+1}'(x)p_n(x) - p_{n+1}(x)p_n'(x)}{\lambda_1\cdots\lambda_n}$$

Pf) Multiply $p_n(y)$ to $\circledast$.

$$p_{n+1}(x)p_n(y) = (x-\cancel{b_n})p_n(x)p_n(y) - \lambda_n p_n(y)p_{n-1}(x)$$

Interchange x, y:

$$p_{n+1}(y)p_n(x) = (y-\cancel{b_n})p_n(x)p_n(y) - \lambda_n p_n(x)p_{n-1}(y).$$

Subtract:

$$p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)$$
$$= (x-y)p_n(x)p_n(y) + \lambda_n\left(p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)\right)$$

let $f_n = p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)$.

$$(x-y)p_k(x)p_k(y) = f_k - \lambda_k f_{k-1}. \quad (f_{-1}=0)$$

Divide by $\lambda_1\cdots\lambda_k(x-y)$

$$\frac{p_k(x)p_k(y)}{\lambda_1\cdots\lambda_k} = \frac{f_k}{\lambda_1\cdots\lambda_k(x-y)} - \frac{f_{k-1}}{\lambda_1\cdots\lambda_{k-1}(x-y)}$$

Summing over $k=0,\dots n$, we get ①.

Write ① as

$$\sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{\lambda_1\cdots\lambda_k} = \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(y) + p_{n+1}(y)p_n(y) - p_{n+1}(y)p_n(x)}{\lambda_1\cdots\lambda_n(x-y)}$$

Taking $\lim_{y\to x}$ we get ②. □

<u>lem</u> $\mathcal{L}$: positive-definite lin ftnl
with monic OPs $\{P_n(x)\}$.
$\Rightarrow$ $P_n(x)$ has n distinct real roots
$\forall n \geq 1$

Pf). Since $\mathcal{L}(P_n(x)) = 0$,
$P_n(x)$ must have a root of odd multiplicity.

($\because$ otherwise $P_n(x) \geq 0$. $\forall x$.
$\Rightarrow$ $\mathcal{L}(P_n(x)) > 0$, contradiction).

let $x_1, \ldots x_k$ be the distinct real zeros.
of $P_n(x)$ with odd multi.
Then

$$\mathcal{L}\Big(\underbrace{(x-x_1)\cdots(x-x_k)}_{\deg k} P_n(x)\Big) > 0$$

By orthogonality $k = n$.
$\Rightarrow$ $P_n(x)$ has n dist real roots. ◻

Thm. $\mathcal{L}$: positive-definite lin ftnl with monic OPS $\{P_n(x)\}$.

$\Rightarrow$ $P_n(x)$ has n distinct real roots

$$x_{n,n} < x_{n,n-1} < \cdots < x_{n,1}$$

with interlacing property

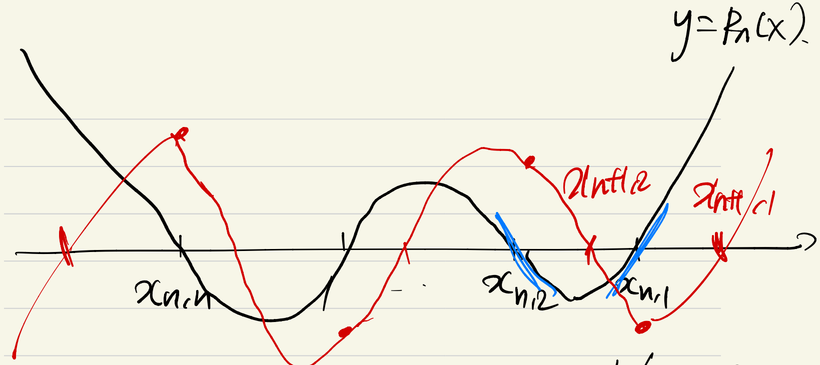
$$x_{n+1,n+1} < x_{n,n} < x_{n+1,n} < x_{n,n-1} < \cdots$$

$$\cdots \quad x_{n+1,2} < x_{n,1} < x_{n+1,1}.$$

Pf) By C-D id ② with $x = x_{n,j}$

$$0 < \sum_{k=0}^{n} \frac{P_k(x_{n,j})^2}{\lambda_1 \cdots \lambda_k} = \frac{P'_{n+1}(x_{n,j})\cancelto{0}{P_n(x_{n,j})} - P_{n+1}(x_{n,j})P'_n(x_{n,j})}{\lambda_1 \cdots \lambda_n}.$$

$\Rightarrow$ $P_{n+1}(x_{n,j})$ has the opposite sign of $P_n'(x_{n,j})$.



$P_n'(x_{n,1}) > 0$

sign of $P_n'(x_{n,j}) = (-1)^{j-1}$

" $\quad P_{n+1}(x_{n,j}) = (-1)^{j}$

$\Rightarrow$ one root in $(x_{n,j+1}, x_{n,j})$ $(j=1,\ldots,n-1)$

Since $\lim_{x\to\infty} P_{n+1}(x) = \infty$,

$\lim_{x\to-\infty} P_{n+1}(x) = (-1)^{n+1}\infty$

$\Rightarrow$ interlacing! $\square$

We will focus on combinatorics of OP.

Plan

We review

basics of combinatorics

generating functions

formal power series.

Dyck paths, Motzkin paths.

matchings, set partitions

permutations.

There are 2 ways to study OP.

① general theory.

② special OP.

	moments	comb obj.
Hermite	$\mu_{2n} = (2n-1)!!$	perfect matchings
Charlier	$\mu_n =$	set partitions
Laguerre	$\mu_n = n!$	permutations.