Ch3. Basics of enumerative combinatorics.

Notation: $[n] = \{1, 2, \dots, n\}$.

§3.1. Formal power series and generating functions.

A power series is a series of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

a_n: coefficient of x^n in $f(x)$.

a_0: the constant term of $f(x)$.

If $a_n \in \mathbb{R}$, then $f(x)$ can be considered as a function defined on the set of x for which the series converges.

ex) If $|x| < 1$,

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

But this doesn't make sense if $|x| \geq 1$.

Let R be a commutative ring with 1.

$R[x]$ = the ring of polynomials in x with coeffs in R.

Def) The ring of formal power series in x with coeffs in R is the set

$$R[[x]] = \{a_0 + a_1 x + a_2 x^2 + \cdots : a_0, a_1, a_2, \dots \in R\},$$

with addition

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

multi

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$$

(Formal power series are polynomials with infinite degree.)

The multiplicative identity is $1 = 1 + 0\cdot x + 0\cdot x^2 + \cdots$

The multiplicative inverse of $f(x)$ is a formal power series $g(x)$ such that $f(x)g(x) = 1$. (if $g(x)$ exists)

We write $f(x)^{-1}$ or $\frac{1}{f(x)}$ for inv of $f(x)$.

ex). $1+x+x^2+\cdots = \frac{1}{1-x}$ holds.

because

$(1+x+x^2+\cdots)(1-x)$

$= (1+\cancel{x}+\cancel{x^2}+\cdots) - (\cancel{x}+\cancel{x^2}+\cdots)$

$= 1.$

$\Rightarrow (1+x+x^2+\cdots)^{-1} = 1-x.$

and $(1-x)^{-1} = 1+x+x^2+\cdots$

$\qquad \| $

$\quad \frac{1}{1-x}$

Important aspect of formal power series

: the coeff of x^n must be computed in a finite number of add & mult.

ex) $e^{1+x} := \sum_{n\geq 0} \frac{(1+x)^n}{n!}$

is not a formal power series in $\mathbb{R}[[x]]$.

The constant term is $\sum_{n\geq 0} \frac{1}{n!} \rightarrow$ infinite sum.

ex) $e\cdot e^x := e \sum_{n\geq 0} \frac{x^n}{n!}$

is a formal power series in $\mathbb{R}[[x]]$.

because coeff of x^n is $\frac{e}{n!}$.

For two formal power series

$f(x) = \sum_{n\geq 0} f_n x^n, \quad g(x) = \sum_{n\geq 0} g_n x^n \quad (g_0=0)$

we define the composition of f and g by

$f(g(x)) = \sum_{n\geq 0} f_n\, g(x)^n.$

Since $g_0=0$, $g(x)^n$ has lowest degree $\geq n$.

The coeff of x^m in $f(g(x))$ is equal to coeff of x^m in $\sum_{n=0}^{m} f_n g(x)^n$

If $g_0 \neq 0$, const term of $f(g(x))$ is $\sum_{n=0}^{\infty} f_n g_0$

Prop R: field. $f(x) \in R[[x]]$.

$f(x)^{-1}$ exists $\Longleftrightarrow$ $\underbrace{f(0)}_{\text{constant term.}} \neq 0$.

Pf) ($\Rightarrow$) Let $g(x) = f(x)^{-1}$. Suppose $f(0) = 0$.
The const term of $f(x)g(x)$ is $f(0)g(0) = 0$
But $f(x)g(x) = 1$ has const term 1
Contradiction.

($\Leftarrow$) Let's write

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$$

$$= f_0 \left(1 + f_0^{-1} f_1 x + f_0^{-1} f_2 x^2 + \cdots\right)$$

$$= f_0 \left(1 - h(x)\right).$$

$$h(x) = \sum_{n \geq 1} h_n x^n, \quad h_n = -f_0^{-1} f_n.$$

$$\frac{1}{f(x)} = \frac{1}{f_0} \cdot \frac{1}{1 - h(x)} = \frac{1}{f_0}\left(1 + h(x) + h(x)^2 + \cdots\right)$$

$h(x)^n$ has lowest deg $\geq n$. $\square$.

As in calculus we define the <u>derivative</u> of

$$f(x) = \sum_{n \geq 0} f_n x^n$$

by $$f'(x) = \sum_{n \geq 1} n f_n x^{n-1}.$$

Prop

$$\left(f(x) g(x)\right)' = f'(x) g(x) + f(x) g'(x)$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) g(x) - f(x) g'(x)}{g(x)^2} \qquad (g(0) \neq 0)$$

$$f(g(x))' = f'(g(x))\, g'(x). \qquad (g(0) = 0.)$$

Def) The generating function for a seq $\{a_n\}_{n\geq 0}$ is the formal power series

$$\sum_{n\geq 0} a_n x^n.$$

ex). The gen. ftn. for $a_n = 2^n$ is

$$\sum_{n\geq 0} 2^n x^n = \sum_{n\geq 0} (2x)^n = \frac{1}{1-2x}.$$

ex) $a_n = n \cdot 2^n$.

Take derivative in prev ex.

$$\sum_{n\geq 0} n 2^n x^{n-1} = \frac{2}{(1-2x)^2}$$

Mult by x

$$\sum_{n\geq 0} n 2^n x^n = \frac{2x}{(1-2x)^2}.$$

Def) A : a set of (combinatorial) objects.

A weight of A is a function

$$wt : A \longrightarrow \underbrace{R}_{\text{any ring}}.$$

The generating function for A with respect to wt is

$$\sum_{a\in A} wt(a).$$ (provided this sum is valid)

ex). $A = \{0, 1, 2, \dots\}$.

$wt(a) = x^a$.

the gen ftn for A is

$$\sum_{a\in A} wt(a) = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}.$$

ex). A : set of all subsets of $[n]$.

$$wt(a) = x^{|a|} y^{n-|a|}$$

The gen ftn for A is

$$\sum_{a \in A} wt(a) = \sum_{a \subseteq [n]} x^{|a|} y^{n-|a|}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$$

ex) A = set of permutations of $[n]$.

$$wt(a) = x^{\text{cycle}(a)}$$

$\text{cycle}(a) = \#$ cycles in a.

$$\Rightarrow \sum_{a \in A} wt(a) = x(x+1) \cdots (x+n-1).$$