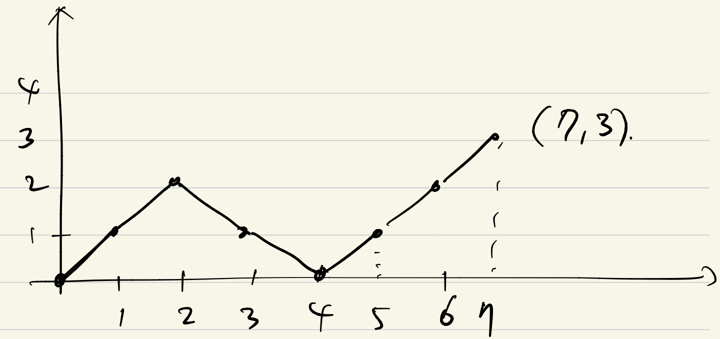
§3.2. Dyck paths and Motzkin paths

Def) A <u>lattice path</u> from u to v is a sequence $(v_0, v_1, \dots, v_n)$ of points in $\mathbb{Z} \times \mathbb{Z}$ with $v_0 = u$, $v_n = v$.

Each pair (v_i, v_{i+1}) is called a <u>step</u>.

sometimes $(v_i, v_{i+1}) \underset{\text{identify}}{=} v_{i+1} - v_i \in \mathbb{Z} \times \mathbb{Z}$.

Def) A <u>Dyck path</u> is a lattice path consisting of <u>up steps</u> $(1,1)$ and <u>down steps</u> $(1,-1)$ staying weakly above x-axis.



$\mathrm{Dyck}(u \to v)$ = set of all Dyck paths from u to v.

$$\mathrm{Dyck}_{2n} = \mathrm{Dyck}((0,0) \to (2n,0)).$$

Q: $|\mathrm{Dyck}_{2n}| = ?$

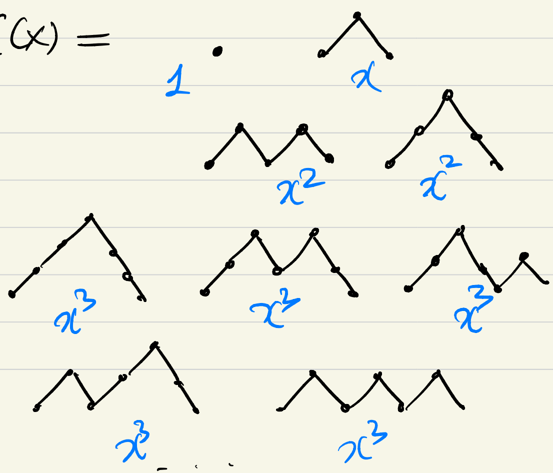
Let $C(x) = \sum_{n\geq 0} |\mathrm{Dyck}_{2n}|\, x^n$.

$\Rightarrow C(x) = \sum_{\pi \in \mathrm{Dyck}} wt(\pi)$

$\pi \in \underbrace{\mathrm{Dyck}}$ all Dyck paths from $(0,0)$ to $(2n,0)$ for $n \in \mathbb{Z}_{\geq 0}$

$wt(\pi) = x^{(\#\text{down steps in } \pi)}$

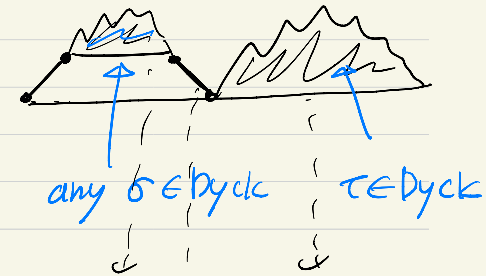
$C(x) =$



$C(x) = 1 + x + x^2 + x^2 + x^3 + \dots + x^3 + \dots$

$= 1 + x + 2x^2 + 5x^3 + \dots$

$C(x) = \underset{\bullet}{1} +$



$= 1 + C(x) \cdot x \cdot C(x).$

$\Rightarrow xC^2 - C + 1 = 0.$

$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$ — : correct sign.

const term $= C(0) = \frac{1 \pm 1}{2 \cdot 0}$

$C(0) = 1$

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}}$$

$$= \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4x)^n$$

$$= \sum_{n\geq 0} \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!} (-1)^n 4^n x^n$$

$$= 1 - \sum_{n\geq 1} \frac{1\cdot 3\cdots(2n-1)}{n!} 2^n x^n$$

$(n-1)!$

$$= 1 - \sum_{n\geq 1} \frac{(2n-2)!}{n!\,(n-1)!} \cdot 2 \cdot x^n$$

$$= 1 - 2\cdot \sum_{n\geq 1} \binom{2n-2}{n-1} \cdot \frac{1}{n} x^n$$

<u>Binomial thm</u>

$$(1+x)^\alpha = \sum_{n\geq 0} \binom{\alpha}{n} x^n$$

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$

$$\Rightarrow C(x) = \sum_{n\geq 1} \frac{1}{n} \binom{2n-2}{n-1} x^{n-1}$$

$$\sum |\mathrm{Dyck}_{2n}| x^n = \sum_{n\geq 0} \frac{1}{n+1} \binom{2n}{n} x^n$$

$$\longrightarrow |\mathrm{Dyck}_{2n}| = \frac{1}{n+1}\binom{2n}{n}.$$

The <u>Catalan number</u> is

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

$C_0, C_1, \ldots$

$= 1, 1, 2, 5, 14, 42, 132, 429, \ldots$

Stanley collected > 200
"Catalan objects".

Some Catalan objects.

① Dyck paths of len $2n$

② ballot seq. of len $2n$

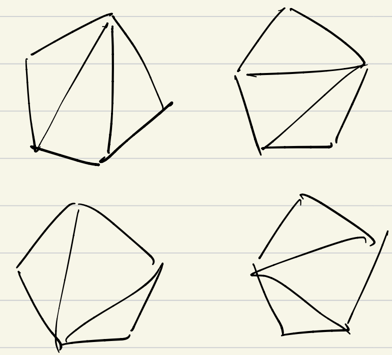
③ triangulation of $(n+2)$-gon

④ plane binary trees with n vertices.

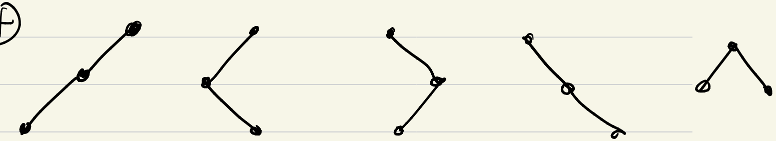
$n=3$

② : AAABBB, AABABB ...

③



④



<u>Prop</u> If $n \geq 1$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k} \qquad (C_0 = 1)$$

$C_1 = C_0 C_0 = 1$

$C_2 = C_1 C_0 + C_0 C_1 = 2$

$C_3 = C_2 C_0 + C_1 C_1 + C_0 C_2 = 2 + 1 + 2 = 5.$

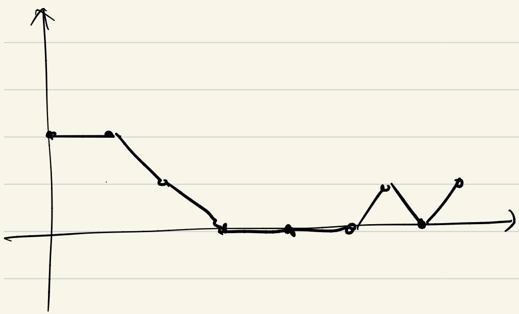
Def) A <u>Motzkin path</u> is a lattice path consisting of

up steps $(1,1)$

down " $(1,-1)$

horizontal " $(1,0)$

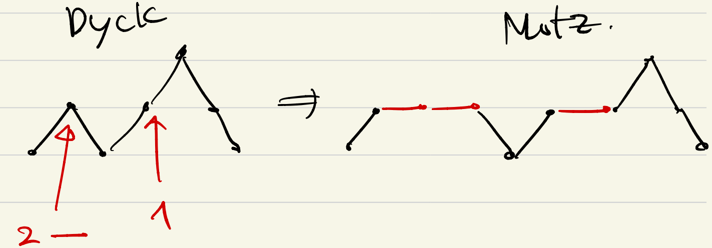
staying weakly above x-axis.



$\mathrm{Motz}(u \to v)$ = the set of all Motzkin paths from u to v.

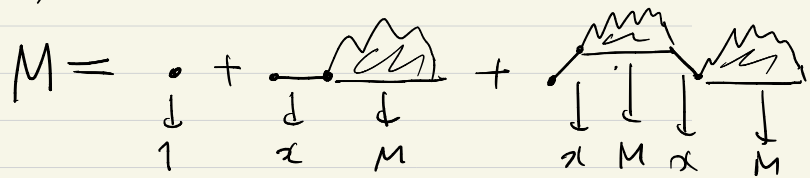
$\mathrm{Motz}_n = \mathrm{Motz}((0,0) \to (n,0))$.

<u>Prop</u> $\displaystyle |\mathrm{Motz}_n| = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k.$



<u>Prop</u> $\displaystyle \sum_{n \geq 0} |\mathrm{Motz}_n| x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$

Pf) let LHS $= M(x)$.



$M = 1 + xM + x^2M^2.$

§3.3. Set partitions and matchings.

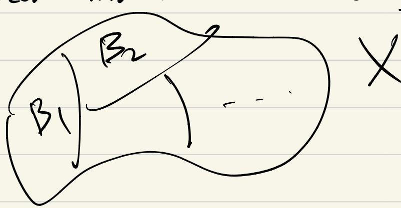
Def) A <u>set partition</u> of a set X is a collection $\pi = \{B_1, \dots, B_k\}$ of subsets of X satisfying

① $B_i \neq \emptyset \quad \forall i$

② $B_i \cap B_j = \emptyset \quad \forall i \neq j$

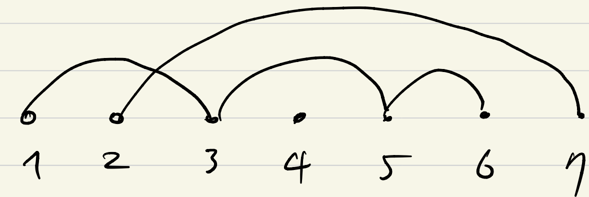
③ $B_1 \cup \dots \cup B_k = X$.

Each B_i is called a <u>block</u>.



A set partition of $[n] = \{1, \dots, n\}$ can be visualied.

e.g. $\pi = \{\{1,3,5,6\}, \{2,7\}, \{4\}\}$



<u>Def</u> Π_n = set of all set partitions of $[n]$.

$\Pi_0 = \{\emptyset\}$.

Def). $\Pi_{n,k}$ = set of all set partitions of $[n]$ with k blocks.

Def) The <u>Stirling number of 2nd kind</u> is $S(n,k) = |\Pi_{n,k}|$.

ex) $S(0,k)=\delta_{k,0}$

$S(n,0)=\delta_{n,0}$

$S(n,n)=1$

$S(n,k)=0$ if $k>n$

Prop For $n,k\geq 1$,

$$S(n,k)=S(n-1,k-1)+k\,S(n-1,k)$$

pf) $\pi\in\Pi_{n,k}$

case I $\{n\}\in\pi$.

$\Rightarrow$ π with $\{n\}$ removed $\in\Pi_{n-1,k-1}$.

$\longrightarrow S(n-1,k-1)$

case II $\{n\}\notin\pi$.

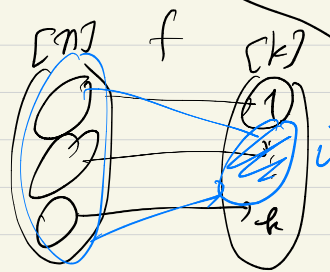
(π with n deleted) $\in\Pi_{n-1,k}$

$\Rightarrow$ $k\cdot S(n-1,k)$ possible such set partitions. $\square$

Prop $S(n,k)=\dfrac{1}{k!}\displaystyle\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n$.

pf) # onto functions $f:[n]\to[k]$

$= k!\,S(n,k)$



by principle of inclusion & exclusion

$$=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}i^n$$

$\square$

Def) A *falling factorial* is

$$(x)_n := x(x-1)\cdots(x-n+1)$$

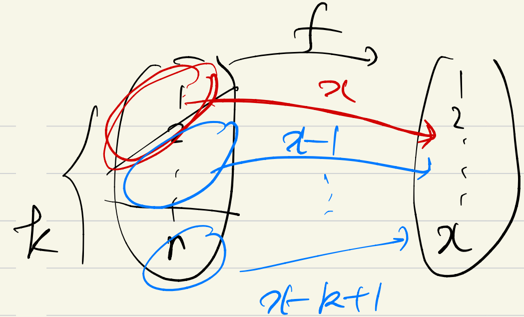
Prop $\displaystyle\sum_{k=0}^{n} S(n,k)(x)_k = x^n.$

Pf) We may assume x is a positive integer.

$x^n = \#$ functions $f : [n] \to [x]$.

$$= \sum_{k=0}^{n} \left(\# f : [n] \to [x] \text{ with } k \text{ elts in the image of } f \right)$$

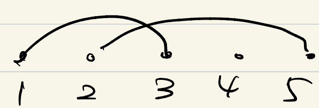
$$= \sum_{k=0}^{n} \underbrace{S(n,k)}\,(x)_k .$$



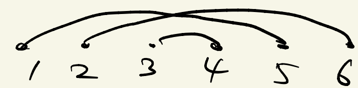
Def) A *matching* on a set X is a set partition of X with blocks of size 1 or 2.
A block of size 1 is called a *fixed point*.
" 2 " an *edge* or an *arc*.

If there are only blocks of size 2, then it is called a *perfect matching* (or *complete* ").

ex.



: 2 edges
: 1 fixed pt.



: perfect matching.

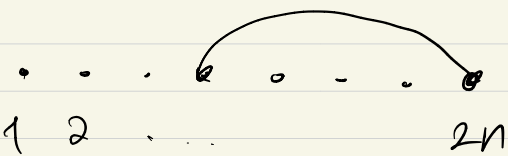
Prop # complete matchings on $[2n]$

is $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)$.

matchings on $[n]$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (2k-1)!!$$

Pf)



ways to connect $2n$ with an integer $= 2n-1$

ways to connect max remaining integer $= 2n-3$

$\vdots$

$\Rightarrow (2n-1)(2n-3) \cdots 3 \cdot 1$.