§3.4. Permutations

Def) A <u>permutation</u> on $[n]=\{1,\dots,n\}$ is a bijection $\pi:[n]\to[n]$.

The <u>symmetric group</u> $\mathfrak{S}_n$ is the group of permutations on $[n]$ with multiplication $\pi\sigma := \pi\circ\sigma$
composition of ftns.

$$(\pi\sigma)(i)=\pi(\sigma(i))$$

We will write

$$\pi = \underbrace{\pi_1\pi_2\cdots\pi_n}_{\text{one-line notation}}, \quad \pi_i=\pi(i)$$

two-line notation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$$

ex). $\pi\in\mathfrak{S}_3$, $\pi(1)=2$
$\pi(2)=3$
$\pi(3)=1$

$$\pi=\pi_1\pi_2\pi_3=231$$

$$=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\pi\pi=\pi^2=\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}=312$$

A <u>cycle</u> of π is a sequence $(a_1,\dots,a_k)$ of distinct elts such that

$\pi(a_1)=a_2$,
$\pi(a_2)=a_3$
$\vdots$
$\pi(a_k)=a_1$

We consider $(a_1,\dots,a_k)$
$=(a_j,\dots,a_k,a_1,\dots,a_{j-1})$

A cycle $\rho = (a_1, \dots, a_k)$ is also considered as a perm in S_n s.t. (length k)

$$\rho(a_i) = a_{i+1} \quad (a_{k+1} = a_1).$$

$$\rho(r) = r \quad (\text{if } r \neq a_1, \dots, a_k)$$

Def) A transposition is a cycle of length 2.

$\hookrightarrow (i, j)$

A simple transposition is a cycle of this form $(i, i+1)$.

let $\pi = \pi_1 \cdots \pi_n \in S_n$, $\tau = (i, j)$.

$$\pi\tau = \begin{array}{ccccccccccc} 1 & \cdots & & i & & \cdots & & j & & \cdots & n \\ \pi_1 & \cdots & \pi_{i-1} & \pi_j & \pi_{i+1} & \cdots & \pi_{j-1} & \pi_i & \pi_{j+1} & \cdots & \pi_n \end{array}$$

$\tau\pi = \pi$ with i & j interchanged.

$(\because$ If $\pi = \cdots i \cdots j \cdots \Rightarrow \tau\pi = \cdots j \cdots i \cdots)$

Prop let $\pi \in \mathfrak{S}_n = S_n$.

$\Rightarrow \pi = \rho_1 \cdots \rho_k$ for some disjoint cycles $\rho_1, \dots, \rho_k \in \mathfrak{S}_n$.

Moreover, $\pi = \tau_1 \cdots \tau_r$, τ_i : simple trans.

Pf) Take $m = 1$.

$$m, \pi(m), \pi^2(m), \pi^3(m), \dots \in [n]$$

$\Rightarrow \pi^i(m) = \pi^j(m)$ for some $i < j$

$\Rightarrow \pi^0(m) = m = \pi^{j-i}(m)$.

$\Rightarrow \exists$ smallest $r \geq 1$, $\pi^r(m) = m$.

$\Rightarrow \underbrace{(m, \pi(m), \dots, \pi^{r-1}(m))}_{=\rho_1}$: cycle of π.

Repeat with $m = \min$ of $[n] \setminus \rho_1$.

$\Rightarrow \pi = \rho_1 \cdots \rho_k$.

$\pi(i,i+1) = \cdots \pi_{i+1}\pi_i \cdots$

So we can sort

$\pi \longrightarrow 1\,2 \cdots n$

$\pi = \tau_1 \cdots \tau_r$, τ_i: sim. tr.

$\square$

In cycle notation,

$\pi = \rho_1 \cdots \rho_k$, ρ_i's: disjoint cycles.

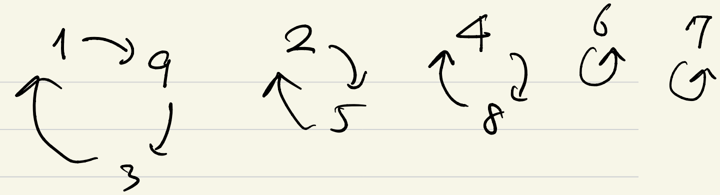
Def) $\mathrm{cycle}(\pi) = \#$ cycles in π.

ex). $\pi = 9\,5\,1\,8\,2\,6\,7\,4\,3 \in \mathfrak{S}_9$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 5 & 1 & 8 & 2 & 6 & 7 & 4 & 3 \end{pmatrix}$$

$= (1,9,3)(2,5)(4,8)(6)(7) \leftarrow$ cycle notation

$= (1,9,3)(2,5)(4,8)$



$\mathrm{cycle}(\pi) = 5$.

Def) A permutation $\pi \in S_n$ is an <u>Involution</u> if $\pi^2 = id$.

Every Involution has cycles of len 1 or 2.

$\Updownarrow$

bijection between Involutions on $[n]$ & matchings on $[n]$.

Def) $\pi \in S_n$.

An <u>inversion</u> of π is a pair (i,j) such that $i<j$, $\pi_i > \pi_j$.

$\mathrm{inv}(\pi) = \#$ inversions in π.

ex) $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$

Inversions : $(1,2), (1,4)$
$(3,4)$,

$\mathrm{inv}(\pi) = 3$.

Def) The <u>sign</u> of $\pi \in S_n$ is

$$\mathrm{sgn}(\pi) = (-1)^{\mathrm{inv}(\pi)}$$

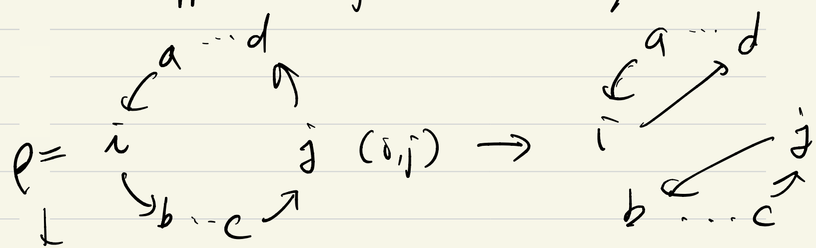
$\mathrm{sgn}(31425) = (-1)^3 = -1$.

$\mathrm{sgn}(id) = 1$.

<u>Lem</u> $\pi \in S_n$, $\tau = (i,j) \in S_n$.

$$\mathrm{cycle}(\pi\tau) = \mathrm{cycle}(\tau\pi) = \begin{cases} \mathrm{cycle}(\pi) - 1 & \text{if } i,j \text{ in diff cycles in } \pi \\ \mathrm{cycle}(\pi) + 1 & \text{if } i,j \text{ in same cycle} \end{cases}$$

Pf) Suppose, i,j in same cycle.



a cycle of π.

$\rho\tau$

Lem $\pi \in S_n$, $\tau = (i, i+1) \in S_n$.

$\operatorname{sgn}(\pi\tau) = -\operatorname{sgn}(\pi)$.

Pf)

$$\pi\tau = \begin{pmatrix} \cdots & i & i+1 & \cdots \\ \cdots & \pi_{i+1} & \pi_i & \cdots \end{pmatrix}$$

$\operatorname{Inv}(\pi\tau) = \operatorname{Inv}(\pi) \pm 1$

Lem $\pi = \tau_1 \cdots \tau_k$, τ_i : sim trans

$\Rightarrow \operatorname{sgn}(\pi) = (-1)^k$.

Pf)

$$\begin{aligned} \operatorname{sgn}(\pi) &= \operatorname{sgn}(id\, \tau_1 \cdots \tau_k) \\ &= -\operatorname{sgn}(id\, \tau_1 \cdots \tau_{k-1}) \\ &\;\;\vdots \\ &= (-1)^k \operatorname{sgn}(id) = (-1)^k \end{aligned}$$

$\square$.

cor) $\operatorname{sgn}(\pi\sigma) = \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)$.

Pf) let $\pi = \underbrace{t_1 \cdots t_k}_{\text{simple}}$, $\sigma = \underbrace{s_1 \cdots s_r}_{\text{simple}}$

$\Rightarrow \operatorname{sgn}(\pi) = (-1)^k$, $\operatorname{sgn}(\sigma) = (-1)^r$.

$\operatorname{sgn}(\pi\sigma) = (-1)^{k+r}$ $\square$.

Prop $\pi \in S_n$.

$$\mathrm{sgn}(\pi) = (-1)^{\mathrm{inv}(\pi)} = (-1)^{n-\mathrm{cycle}(\pi)}$$

$$= (-1)^{\#\text{ even cycles in } \pi}$$

In particular, if $\pi = t_1 \cdots t_k$,

t_i : transposition

then $\mathrm{sgn}(\pi) = (-1)^k$.

Pf) Let $\pi = t_1 \cdots t_k$, t_i : simple trans.

$$\mathrm{sgn}(\pi) = (-1)^k$$

Since $\pi = t_1 \cdots t_k\, id$

$$(-1)^{\mathrm{cycle}(\pi)} = (-1)^{\mathrm{cycle}(id)+k}$$

$$= (-1)^{n+k}$$

$$\Rightarrow (-1)^k = (-1)^{n-\mathrm{cycle}(\pi)}$$

$$\mathrm{sgn}(\pi) = \mathrm{sgn}(t_1) \cdots \mathrm{sgn}(t_k)$$

$$= (-1) \cdots (-1) = (-1)^k.$$

Let c_i be # cycles of len i.

$$n = 1 \cdot c_1 + 2c_2 + \cdots + nc_n$$

$$c_1 + \cdots + c_n = \mathrm{cycle}(\pi).$$

$$(-1)^{n-\mathrm{cycle}(\pi)} = (-1)^{(1c_1 + 2c_2 + \cdots + nc_n) - (c_1 + \cdots + c_n)}$$

$$= (-1)^{0 \cdot c_1 + 1 \cdot c_2 + \cdots + (n-1)c_n}$$

$$= (-1)^{c_2 + c_4 + \cdots} = (-1)^{\#\text{ even cycles}}.$$

$\square$.

Def) The signless Stirling number of the 1st kind $c(n,k)$ is # permutations on $[n]$ with k cycles.

The Stirling number of the 1st kind is $s(n,k) = \underbrace{(-1)^{n-k}}_{}\, c(n,k)$.

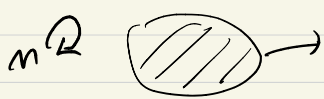
↳ sign of any $\pi \in S_n$ with k cycles.

Prop For $n, k \geq 1$,

$$c(n,k) = c(n-1,k-1) + (n-1)\,c(n-1,k).$$

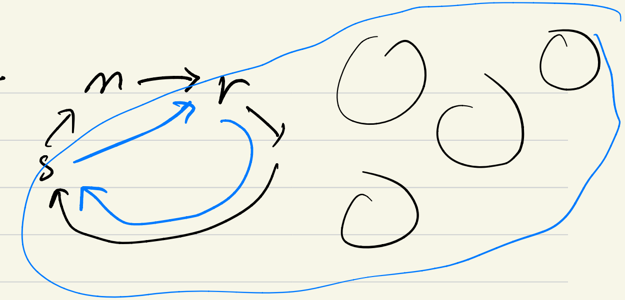
Pf) Let $\pi \in S_n$, $\mathrm{cycle}(\pi) = k$.

case I



$\to c(n-1,k-1)$

case II



$c(n-1,k)$

$\Rightarrow \underbrace{(n-1)}_{}\, c(n-1,k)$

ways to insert n.

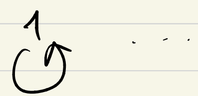
Prop $$\sum_{k=0}^{n} c(n,k)\,x^k = x(x+1)\cdots(x+n-1)$$

Pf) Induction using recursion.

Prop $\displaystyle\sum_{\pi\in S_n} x^{\mathrm{cycle}(\pi)} = x(x+1)\cdots(x+n-1) = x(x+1)(x+1+1)\cdots(x+\underbrace{1+\cdots+1}_{n-1})$

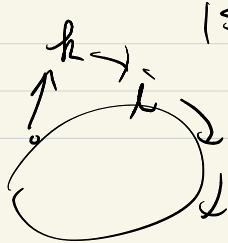
bijective pf

We have an algorithm to construct $\pi\in S_n$.



perm π in S_{k-1}

add k
- create new cycle $k\circlearrowleft$
- insert k before i, $1\le i\le k-1$



$(x+\underbrace{1+\cdots+1}_{k-1})$

$\square$

e.g. $x(x+1)(x+1_a+1_b)$

$= x\cdot x\cdot x + \underbrace{x\cdot x}\cdot 1_a + \underbrace{x\cdot x}\cdot 1_b + \cdots$

Cor $\sum_{k=0}^{n} s(n,k)x^k = (x)_n$.

Pf) $\sum_{k=0}^{n} c(n,k)x^k = x(x+1)\cdots(x+n-1)$

$x \mapsto -x$

multiply $(-1)^n$

Recall $\sum_{k=0}^{n} S(n,k)(x)_k = x^n$

Matrix equation

$$\big(s(n,k)\big)_{n,k\geq 0}\big(x^n\big)_{n\geq 0} = \big((x)_n\big)_{n\geq 0}$$

$$\big(S(n,k)\big)_{n,k\geq 0}\big((x)_n\big)_{n\geq 0} = \big(x^n\big)_{n\geq 0}.$$

Prop

$$\big(s(n,k)\big)\big(S(n,k)\big) = \big(S(n,k)\big)\big(s(n,k)\big) = I$$

$$\sum_{k\geq 0} S(n,k)s(k,m) = \delta_{m,n}$$

$$\sum_{k\geq 0} s(n,k)S(k,m) = \delta_{m,n}.$$