Ch4. Combinatorial model for OPS

§4.1. Orthogonal polynomials and 3-term recurrence.

K : a field (or a commutative ring if division is not used.)

$K[x]$ = ring of polys in x with coeffs in K.

A <u>linear function</u> is a linear map

$\mathcal{L} : K[x] \to K$.

i.e. $\mathcal{L}(af(x) + bg(x)) = a\mathcal{L}(f(x)) + b\mathcal{L}(g(x))$.

The nth <u>moment</u> of $\mathcal{L}$ is

$\mu_n = \mathcal{L}(x^n)$.

Def) $\mathcal{L}$: lin ftnl.

A seq of polys $\{p_n(x)\}_{n\geq 0}$ is OPS for $\mathcal{L}$ if

① $\deg p_n(x) = n$, $\forall n \geq 0$

② $\mathcal{L}(p_m(x)p_n(x)) = 0$ $\forall m \neq n$,

③ $\mathcal{L}(p_n(x)^2) \neq 0$ $\forall n \geq 0$.

We also say $\{p_n(x)\}_{n\geq 0}$ is OPS for $\{\mu_n\}_{n\geq 0}$ and μ_n is moment of OPS $\{p_n(x)\}_{n\geq 0}$.

(K can be any field.
Sometimes such OPS is called
general OPS or
formal OPS .).

Prop Suppose $\{p_n(x)\}_{n\geq 0}$ is OPS for $\mathcal{L}$.

① $\{p_n(x)\}_{n\geq 0}$ is OPS for $\mathcal{L}' = a\mathcal{L}$ $(a\neq 0)$

② $\mathcal{L}$ is uniquely determined up to scalar multiplication.

③ $\{a_n p_n(x)\}_{n\geq 0}$ is OPS for $\mathcal{L}$ $\forall a_n \neq 0$.

From now on we always assume.

① $\deg p_n(x) = n$

② $\mathcal{L}(1) = 1$

③ $p_n(x)$ is monic.

Thm $\{p_n(x)\}_{n\geq 0}$ is OPS for $\mathcal{L}$.

$\Rightarrow \exists \{b_n\}_{n\geq 0}, \{\lambda_n\}_{n\geq 1}$ such that $\lambda_n \neq 0$ and

(*) $p_{n+1}(x) = (x-b_n)p_n(x) - \lambda_n p_{n-1}(x) \quad \forall n\geq 0$

$p_{-1}(x) = 0,\ p_0(x) = 1.$

Thm (Favard's thm).

If $\{p_n(x)\}_{n\geq 0}$ satisfies (*) then it is OPS for some $\mathcal{L}$.

Goal: Find combinatorial models for $p_n(x)$ and μ_n.

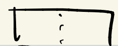
And prove Favard's thm.

§4.2. A model for OP using Favard tilings.

Def). A <u>Favard tiling</u> of size n is a tiling of a $1\times n$ square board with 3 types of tiles:

① red monomino 

② black 〃 

③ black domino 

FT_n = set of all Favard tilings of size n.

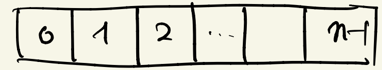
For $T \in FT_n$, define

$$\mathrm{wt}(T) = \prod_{t\in T} \mathrm{wt}(t),$$

$$\mathrm{wt}(\square) = x$$

$$\mathrm{wt}(\boxed{i}) = -b_i$$

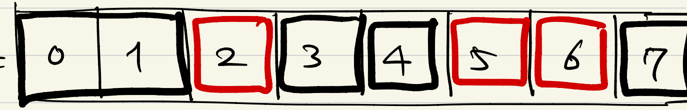
$$\mathrm{wt}(\boxed{i-1 \,\vdots\, i}) = -\lambda_i .$$

$1\times n$ board : 

ex). $n=8$

FT_8

$\rotatebox{90}{$\in$}$

$T=$ 

$$\mathrm{wt}(T) = x^3 b_3 b_4 b_7 \lambda_1$$

<u>Thm</u> Suppose $\{p_n(x)\}$ satisfies

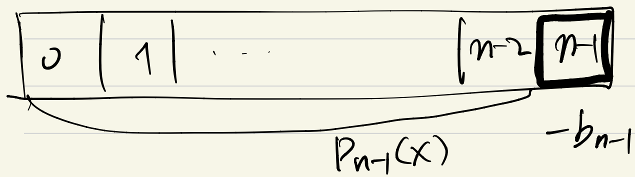
$$p_{n+1}(x) = (x-b_n)p_n(x) - \lambda_n p_{n-1}(x) \qquad \forall n\geq 0$$

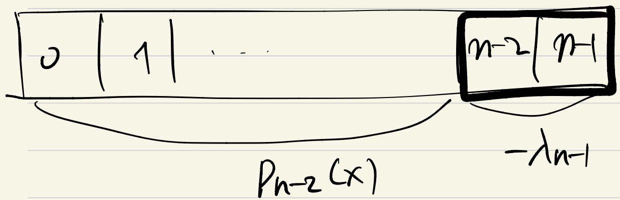
$p_{-1}(x)=0,\ p_0(x)=1.$

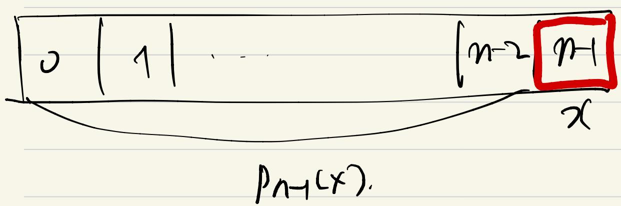
$$\Rightarrow\ p_n(x) = \sum_{T\in FT_n} \mathrm{wt}(T)$$

Pf) Easy by induction.

$\boxed{\text{idea}}$ $\displaystyle\sum_{T\in FT_n} wt(T)$







$$\Rightarrow \sum_{T\in FT_n} wt(T) = (x-b_{n-1})P_{n-1}(x) - \lambda_{n-1}P_{n-2}(x) \quad \square$$

§4.3. How to find a combinatorial model for moments.

Note: $\mu_n = \mathcal{L}(x^n)$ is important because they determine $\mathcal{L}$.

Suppose $\{P_n(x)\}_{n\geq 0}$ is OPS for $\mathcal{L}$. Then

$$\mathcal{L}(P_n(x)) = \begin{cases} 0 & \text{if } n\geq 1 \\ 1 & \text{if } n=0 \end{cases}$$

$$\parallel$$

$$\left(\because \mathcal{L}(P_n(x) P_0(x)) = \delta_{n,0} \right).$$

$$P_{n+1}(x) = (x-b_n) P_n(x) - \lambda_n P_{n-1}(x) \quad \forall n\geq 0$$

$$P_0(x) = 1$$

$$P_1(x) = x - b_0$$

$$P_2(x) = (x-b_1) P_1 - \lambda_1 P_0$$

$$= (x-b_1)(x-b_0) - \lambda_1 = \underbrace{x^2 - (b_0+b_1)x + b_0 b_1 - \lambda_1}$$

$$0 = \mathcal{L}(P_1(x)) = \mathcal{L}(x-b_0) = \mu_1 - b_0$$

$$\Rightarrow \mu_1 = b_0.$$

$$0 = \mathcal{L}(P_2(x)) = \mu_2 - (b_0+b_1)\mu_1 + b_0 b_1 - \lambda_1$$

$$\Rightarrow \mu_2 = (b_0+b_1) b_0 - b_0 b_1 + \lambda_1$$

$$= b_0^2 + \lambda_1$$

$$\mu_3 = b_0^3 + 2 b_0 \lambda_1 + b_1 \lambda_1$$

$$\mu_4 = b_0^4 + 3 b_0^2 \lambda_1 + 2 b_0 b_1 \lambda_1 + b_1^2 \lambda_1 + \lambda_1^2 + \lambda_1 \lambda_2$$

$$\mu_5 = \cdots$$

If there is a nice combinatorial model for μ_n, we can hope

$$\mu_n = \sum_{\pi \in A_n} wt(\pi)$$

where $wt(\pi)$ is a monomial in b_i's, λ_i's.

Let's put $b_i = \lambda_i = 1$.

$\Rightarrow \mu_n = |A_n|$.

In the case $b_i = \lambda_i = 1$

$\mu_0 = 1$ $\quad \mu_3 = 4$
$\mu_1 = 1$ $\quad \mu_5 = 9$
$\mu_2 = 2$, $\quad \vdots$

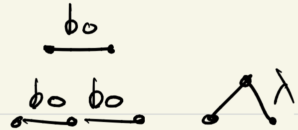
1, 1, 2, 4, 9, 21, 51, 127, 323, ...

https://oeis.com

<u>Guess</u> if $b_i = \lambda_i = 1$

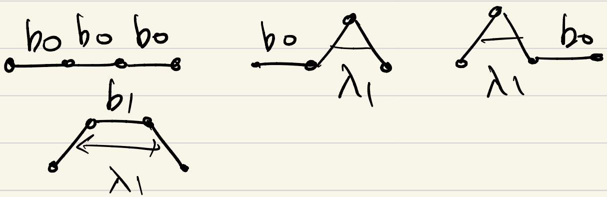
$\mu_n = |Mot_n|$.

$\mu_1 = b_0.$ $\longleftrightarrow$



$\mu_2 = b_0^2 + \lambda_1$

$\mu_3 = b_0^3 + 2b_0\lambda_1 + b_1\lambda_1$

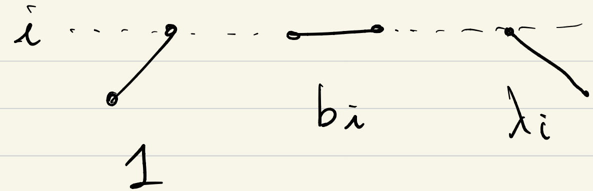


$\mu_4 = b_0^4 + 3b_0^2\lambda_1 + 2b_0b_1\lambda_1 + b_1^2\lambda_1$
$\quad + \lambda_1^2 + \lambda_1\lambda_2$

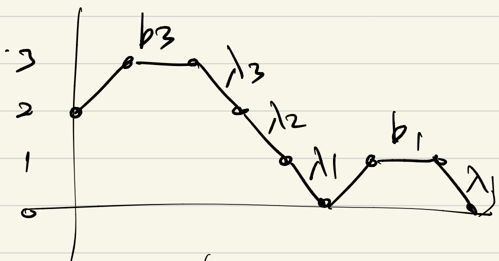
Def). Let π be a Motzkin path.

Define $wt(\pi) = \prod_{\substack{s:\text{step} \\ \text{of } \pi}} wt(s)$

where



e.g.



$\hookrightarrow$ $b_1 b_3 \lambda_1^2 \lambda_2 \lambda_3.$

Thm $\mu_n = \sum_{\pi \in Mot_n} wt(\pi).$

Def). $\{P_n(x)\}_{n\geq 0}$ is OPS for $\mathcal{L}$.

For $n, r, s \geq 0$, the <u>mixed moments</u>

$\mu_{n,r,s}$, $\mu_{n,k}$ of this OPS by

$$\mu_{n,r,s} = \frac{\mathcal{L}(x^n P_r(x) P_s(x))}{\mathcal{L}(P_s(x)^2)}$$

$$\mu_{n,k} = \mu_{n,0,k} = \frac{\mathcal{L}(x^n P_k(x))}{\mathcal{L}(P_k(x)^2)}$$

<u>Note</u> $\mu_n = \mu_{n,0,0} = \mu_{n,0}$

<u>Thm</u>

$$\mu_{n,r,s} = \sum_{\pi \in Mot((0,r)\to(n,s))} wt(\pi)$$